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Mathematical Reviews

Edited by

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J. V. Wehausen, *Executive Editor*

Vol. 15, No. 2 February, 1954 pp. 89–188

The index issue for Vol. 14 will appear as Part 2 of one of the issues of Vol. 15.

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

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Editorial Office

MATHEMATICAL REVIEWS, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

Vol. 15, No. 2 FEBRUARY, 1954 Pages 89-188

HISTORY

*Eves, Howard. **An introduction to the history of mathematics.** Rinehart and Company, Inc., New York, 1953. xv+422 pp. $6.00.

This book deals with the history of elementary mathematics. The first nine chapters do not require more knowledge than that of simple arithmetic and of high school algebra, geometry and trigonometry. For the subsequent chapters some knowledge of analytic geometry and the basic concepts of the calculus is needed. Any concepts or developments of a more advanced nature are explained at the points where they are introduced. A remarkable feature is the inclusion of Problem Studies at the end of each chapter, containing problems associated with the historical subjects treated in the text. These subjects are: Number systems (ch. I); Babylonian and Egyptian mathematics (ch. II); Greek mathematics (ch. III–VI); Hindu and Arabian mathematics (ch. VII); European mathematics before 1600 (ch. VIII) and in the 17th century (ch. IX); Analytic geometry (ch. X); The calculus (ch. XI); Transition to modern mathematics (ch. XII). To the various chapters special bibliographies are annexed, which, together with a general bibliography at the end of the work, open the way to a deeper searching study. *E. J. Dijksterhuis.*

Yamamoto, Susumu. Beitrag zur Euklid-Forschung. Ein Quellenstudium über den finiten Charakter der griechischen Mathematik. Comment. Math. Univ. St. Paul. 1, 59–66 (1953).

Es handelt sich um die Deutung eines (bei Euklid ungebräuchlichen) Futurums in der Ekthesis von Elemente X, Prop. 1. Der philosophische Hintergrund des Problems wird zutreffend beschrieben. Es geht aber zu weit, den griechischen Mathematikern eine "finite Einstellung" zuzuschreiben, oder zu behaupten, Euklid denke "nicht einmal in beliebigem n". Auch greift der Verf. wohl fehl, wenn er den Gebrauch des Futurums daraus erklärt, der hier in Rede stehende "Rest" habe keinen genügend bestimmten Charakter. Das Futurum weist vielmehr zurück auf das vorhergehende "immer wieder", und der Gebrauch einer etwas unbestimmten Terminologie ist wohl dem Umstand zuzuschreiben, dass Euklid nicht verfügt über die Methode der rekurrenten Definition, welche ihm erlaubt hätte, erst den nach n-fachem Abnehmen verbleibenden Rest R_n zu definieren, und dann zu behaupten: "Es gibt ein n, sodass $R_n<C$." *E. W. Beth* (Amsterdam).

Truesdell, C. Notes on the history of the general equations of hydrodynamics. Amer. Math. Monthly 60, 445–458 (1953).

Gorelik, G. S. In memory of A. A. Andronov. Uspehi Fiz. Nauk 49, 449–468 (1 plate) (1953). (Russian)
Biography and list of published works.

Casotti, Maria Walcher. Iacopo Barozzi da Vignola nella storia della prospettiva. Period. Mat. (4) 31, 73–103 (1953).

Finikov, S. P. Sergei Sergeevič Byušgens (on his seventieth birthday). Uspehi Matem. Nauk (N.S.) 8, no. 4(56), 185–192 (1 plate). (Russian)
Biographical sketch and list of published works.

Newman, James R. William Kingdon Clifford. Scientific American 188, no. 2, 78–84 (1953).

Nesteruk, F. Ya. Nicholas Copernicus as a hydrotechnician. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1953, 1341–1349 (1953). (Russian)

Dick, Julius. Nikolaus Kopernikus' De revolutionibus. Wissensch. Ann. 2, 450–458 (1953).

*Eulerus, Leonhardus. **Opera omnia. Series prima. Opera mathematica. Vol. XXIV. Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti.** Edidit C. Carathéodory. Societas Scientiarum Naturalium Helveticae, Bern, 1952. lxii +308 pp.

*Eulerus, Leonhardus. **Opera omnia. Series prima. Opera mathematica. Vol. XXV. Commentationes analyticae ad calculum variationum pertinentes.** Edidit C. Carathéodory. Societas Scientiarum Naturalium Helveticae, Bern, 1952. xxviii+343 pp.

According to Speiser's preface to Vol. 24, these two volumes (24 and 25) complete the publication of Euler's work on algebra and analysis, and four more volumes on geometry will end the first series, i.e., the Opera Mathematica. In addition to providing editorial footnotes, Carathéodory contributed a valuable analytical and historical introduction to Euler's work on the calculus of variations (in Vol. 24, pp. i–lxii). This is supplemented by additional historical remarks in Speiser's 26 page preface to Vol. 25. On pp. lvi–lxii of Vol. 24 there is a classified list of the examples treated by Euler in these two volumes, with references to all sections where they are discussed. At the end of Vol. 25 there is a short index of the names appearing in the two volumes. It is interesting to compare Vol. 24 with the 1744 edition of "Methodus inveniendi lineas curvas . . ." [Bousquet, Lausanne-Geneva]. The pages are larger and somewhat fewer, printed in clear modern type, except that the original first page and title page with their engravings have been reproduced. The figures have been inserted in the appropriate places in the text, whereas in the original they were collected on five folded leaves at the end. *L. M. Graves* (Chicago, Ill.).

Graf, Heinrich. Obituary: Sebastian Finsterwalder. Jber. Deutsch. Math. Verein. 56, 27–31 (1953).

*Gromeka, I. S. **Sobranie sočinenil. [Collected works.]** Izdat. Akad. Nauk SSSR, Moscow, 1952. 296 pp. (1 plate) 23.40 rubles.

In addition to his collected works this volume contains a short biography and an essay on his scientific work by N.

A. Talickih. Ippolit Stepanovič Gromeka was born January 27, 1851 and died October 13, 1889 (dates are old style).

On the fiftieth anniversary of Andreĭ Nikolaevič Kolmogorov. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 181–188 (1 plate) (1953). (Russian)

A short description of Kolmogorov's mathematical activity accompanied by a list of almost all his scientific writing.

Aleksandrov, P. S., and Hinčin, A. Ya. Andreĭ Nikolaevič Kolmogorov (for his fiftieth birthday). Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 177–200 (1 plate) (1953). (Russian)

A sketch of Kolmogorov's scientific work and a nearly complete list of his published work.

Taton, René. La première oeuvre géométrique de Philippe de La Hire. Rev. Hist. Sci. Appl. **6**, 93–111 (1953).

Lorey, Wilhelm. Die Bedeutung von Pierre Simon Laplace (28.3. 1749—5.3. 1827) und Felix Klein (25.4. 1849—22.6. 1925) für die Versicherungsmathematik. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 1, 39–50 (1950).

Persico, E. Leonardo e la fisica. Nuovo Cimento (9) **10**, supplemento, 201–210 (1953).

Černyaev, M. P., Nestorovič, N. M., and Lyapin, N. M. Dmitriĭ Dmitrievič Morduhaĭ-Boltovskoĭ (1876–1952). Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 131–139 (1 plate) (1953). (Russian)

A short biography and list of his published works.

***Feys, Robert. Peano et Burali-Forti, précurseurs de la logique combinatoire.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 70–72. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

Segre, Beniamino. Discorso commemorativo dell'insigne matematico Salvatore Pincherle. Rivista Mat. Univ. Parma **4**, 3–10 (1 plate) (1953).

***Leonida Tonelli, in memoriam.** Arti Grafiche Tornar, Pisa, 1952. 193 pp. (2 plates).

A volume containing various telegrams and letters of condolence, speeches, and obituaries in connection with both the funeral and a later commemoration service for Tonelli. Tonelli was born April 12, 1885 and died March 12, 1946.

FOUNDATIONS

***Lorenzen, Paul. Die ontologische und die operative Auffassung der Logik.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 12–18. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

In contrast to current ontological conceptions of logic, which (i) avoid any reference to actual operations, (ii) derive the rules of operation from ontological knowledge as expressed by existence theorems, and hence (iii) have to start from the acceptance of universal theorems, which are borrowed from traditional logic and mathematics, the author develops an operational conception which (i) creates its own object, independently of scientific tradition, and (ii) avoids any attempt at deriving its rules of operation from something else. Mathematicians, however they may interpret this activity, have to operate in accordance with the rules of certain calculi. This activity creates certain problems, the solution of which is the task of mathematics; logic is that part of mathematics which is concerned with quite arbitrary calculi. Starting from the discussion of a very simple calculus, the author shows, with great lucidity, how by and by the usual concepts of logic present themselves quite naturally, without reference to any presupposed logical knowledge. *E. W. Beth* (Amsterdam).

Lorenzen, Paul. Über die Widerspruchsfreiheit des Unendlichkeitsbegriffes. Studium Generale **5**, 591–594 (1952).

Ridder, J. Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen. IV. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 99–110 (1953).

The author discusses limitations on the algebraic systems with a closure operation presented in Parts II and III [same Proc. **55**, 459–467 (1952); **56**, 1–11 (1953); these Rev. **14**, 1052]. He gives decision procedures for equations in these systems, and shows that, for decision procedures of Parts II and III, the matrices used may be limited to those having a simpler designated class than that specified in the earlier work. *D. Nelson* (Washington, D. C.).

Seki, Setsuya. On the weakened type-logic. (Note on metamathematics. I.) Comment. Math. Univ. St. Paul. **2**, 29–40 (1953). See the note on p. 1139.

To Gentzen's calculus of sequences *LK* [Math. Z. **39**, 176–210 (1934)] a finite hierarchy of types is added on orthodox lines. (i) Gentzen's proof of his Hauptsatz (elimination of cuts) applies unchanged to the author's system (LSK). (ii) The axiom of reducibility is consistent in LSK, in effect because it holds in a model of type theory with a single individual, two properties of individuals, etc. [cf. Gentzen's consistency proof for type theory in Math. Z. **41**, 357–366 (1936)]. [Reviewer's note: The author's proof of (ii) is false since he considers a model with just one element of any given type, while the axioms of abstraction may be used to introduce incompatible properties, e.g., $\hat{x}A$ and $\hat{x}-A$.] *G. Kreisel* (Reading).

***Issmann, S. Une méthode de décision pour certaines formules du calcul des prédicats.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 35–38. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

Decision method for the monadic predicate calculus [cf. H. Behmann, Math. Ann. **86**, 163–229 (1922)]. *G. Kreisel* (Reading).

Martin, Norman M. On completeness of decision element sets. J. Computing Systems **1**, 150–154 (1953).

The author is concerned with the completeness in the logical sense of a set of decision elements of radix 2, order 2. He extends a result of Goodell [same J. **1**, 1–13 (1952), pp. 5–6; these Rev. **14**, 691] and gives a necessary and sufficient condition for completeness in terms of certain standard sets he defines. *H. H. Goldstine* (Princeton, N. J.).

Rose, Alan. A formalization of Sobociński's three-valued implicational propositional calculus. J. Computing Systems 1, 165–168 (1953).

Sobociński has recently formalized a three-valued propositional calculus having two designated truth values, the primitive operations being implication and negation [same J. 1, 23–55 (1952); these Rev. 14, 834]. The author has studied the corresponding implicative system obtained by ignoring the negation operation [C. R. Acad. Sci. Paris 235, 1000–1002 (1952); these Rev. 14, 834]. He has shown that any weakly complete formalization of this system having substitution and modus ponens as its only rules of procedure has its degree of completeness equal to 3. In the present note he gives a formalization of this implicative system which is shown to be both weakly and strongly complete. This formalization has six axioms, some of which are complicated. In addition to substitution and modus ponens it has also the following third rule of procedure: If the formulas P and Q have no variable in common, and CCPCQQCQQ is a theorem, then P is a theorem. *O. Frink.*

Sugihara, Takeo. Negation in many-valued logic. Mem. Lib. Arts Dept. Fukui Univ. 1952, no. 1, 65–69 (1952).

The type of n-valued logic treated here consists of a finite lattice in which there is defined a negation operation p^*, obeying the three laws: (A1) $p \cap p^* = 0$; (A2) $(p \cup q)^* = p^* \cap q^*$; and (A3) $p \leqq p^{**}$. Four special types are distinguished in which negation satisfies also some additional condition, such as a dual of one of the above. Examples of each type are exhibited. Eleven theorems which hold in all such systems are derived. Each of the last four of these theorems actually consists of many theorems combined into one statement. *O. Frink* (State College, Pa.).

***Fitch, Frederic B. Self-referential relations.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 121–127. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

An n-place relation R is called self-referential with regard to an $n+1$-place propositional function Φ if, for all $a_1, \cdots, a_n$, $Ra_1 \cdots a_n$ is provable if and only if $\Phi(R, a_1, \cdots, a_n)$ is provable. The author discusses the problem of finding such an R if Φ is given. He shows that, for a certain formal system K closely related to his Basic Logic [J. Symbolic Logic 7, 105–114 (1942); these Rev. 4, 125], this problem admits of a general solution. Hence we can find a class that has itself as its only member, two classes each of which has the other as its only member, and even a non-terminating sequence of classes each of which has the next member of the sequence as its only member; this sequence can, if we wish, be constructed so as to turn back on itself and so form a "circle". Moreover, the solution is given of a still more general problem: Given an $n+1$-place propositional function Φ and $n+1$-place functions $\Psi_1, \cdots, \Psi_m$, to find an m-place relation R such that, for all $a_1, \cdots, a_n$,

$$R(\Psi_1(R, a_1, \cdots, a_n) \cdots (\Psi_m(R, a_1, \cdots, a_n))$$

is provable if and only if $\Phi(R, a_1, \cdots, a_n)$ is provable. *E. W. Beth* (Amsterdam).

Yuting, Shen. Paradox of the class of all grounded classes. J. Symbolic Logic 18, 114 (1953).

The author calls a class grounded if it is not the first of an infinite sequence of classes (not necessarily distinct) each a member of the preceding. If a class is not grounded, it is called groundless. It follows that if the class of all grounded classes is grounded, then it is groundless, and conversely. Two similar paradoxes are described which concern the class of all non-circular classes, and the class of all not n-circular classes. The Russell class of all classes not members of themselves is the special case $n=1$ of the class of all not n-circular classes. *O. Frink* (State College, Pa.).

***Heyting, A. "Espace de Hilbert et intuitionnisme".** Les méthodes formelles en axiomatique. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 36, Paris, 1950, pp. 59–63; discussion, p. 64. Centre National de la Recherche Scientifique, Paris, 1953.

The first part of this lecture contains a general discussion of the relationship between intuitionism and formalism. It is stressed that in spite of the well-known differences between the two schools of thought, they do in fact have a great deal in common. According to the present author, the metamathematical basis on which Hilbert intended to build his formalistic edifice originally coincides approximately with the intuitionistic theory of numbers. However, whereas Hilbert permits purely formal sentences in the further development of Mathematics, the intuitionist, though agreeing to the provisional use of formal methods for the sake of expediency, insists that it should always be possible to interpret the entire theory from the intuitionistic point of view.

In the second part of his lecture the author gives the intuitionistic interpretation of various concepts and results of classical mathematics. He stresses the importance of the notion of distance in intuitionism (a is distant from b if one can find a rational interval separating a from b). It is shown that if the discriminant of a quadratic form of two variables is positive, then the value of the form is distant from zero whenever at least one of the arguments is distant from zero. Coming to Hilbert space, the author discusses the intuitionistic interpretation of the concept of linear independence. He then points out that while a completely continuous quadratic form always possesses a maximum, it may not be possible to determine a point for which the maximum is attained. However, a condition is given under which the determination of such a point becomes possible.

A. Robinson (Toronto, Ont.).

Carter, W. C., and Rettig, A. S. Analytic minimization methods. I. Conjunctive forms. J. Computing Systems 1, 179–195 (1953).

The purpose of this paper is to explain a new method of reducing sentential combinations to a simplified conjunctive form. This reduction is accomplished by the application of a series of simple rules. These rules do not require that the function be entered on a chart, nor that the distinguished conjunctive or disjunctive normal form be found. Thus the number of sentential variables in the combination is not a limiting factor for this method. In addition, a set of theorems will show that these simplified forms may be chosen to be minimal. (From the introduction.) *S. Sherman.*

Skolem, Th. On the nature of mathematical reasoning. Gaceta Mat. (1) 4, 113–124 (1952). (Spanish)

This begins with a general statement of the principal views as to the nature of mathematics. This statement is critical rather than expository or historical. The author rejects Platonism because its extreme form leads to contradiction; this circumstance, in connection with other objections, the author regards as showing that the Platonist

position is unsound in principle, so that any modification of Platonism is to be regarded with suspicion. He rejects intuitionism because of its subjective character. Finally, although he seems sympathetically inclined towards formalism, yet he regards the fact that the consistency of the stronger system runs into the difficulty of the Gödel theorem as a serious objection. This difficulty does not apply to systems with only free variables, such as the primitive recursive arithmetic founded by the author. Accordingly, he advances the thesis that all use of bound variables over an infinite range ought to be avoided. From this point of view he finds that even the theory of general recursive functions is unsatisfactory because the very definition of general recursiveness involves such an improper quantification. He concludes by an appeal for the use of formal methods with special emphasis on those using only free variables.

H. B. Curry (State College, Pa.).

Skolem, Th. Considerations on the foundations of mathematics. Revista Mat. Hisp.-Amer. (4) **12**, 169–200 (1952); **13**, 149–174 (1953). (Spanish)

According to a letter from the author this paper and the one reviewed above are based on lectures given in the spring of 1952 at the "Consejo superior de investigaciones cientificas" in Madrid. The lectures were given in English and German; they were translated into Spanish by Ramón Fuentes. The paper, after a short historical introduction, expounds the propositional algebra and predicate calculus, with emphasis on the intuitionistic systems, and then concludes with a brief treatment of recursive arithmetic (mostly primitive). In spite of its elementary character it contains some interesting and original material. The treatment of the logical calculus is synthetic in the sense that the operations are introduced one at a time in the following order: implication, conjunction, alternation, negation, free variables, and quantifiers. At each stage new axioms are introduced [essentially those of Hilbert and Bernays, Grundlagen der Mathematik, vol. I, Springer, Berlin, 1934], and an argument is given which aims to show that the extension is conservative in the sense (of Rosenbloom) that no new formulas not containing the new operation are deducible. For the intuitionistic algebra of pure implication it is shown that every formula containing not more than two variables is equivalent to one or the other of 14 special formulas in those variables; when conjunction is introduced there are 8 more such forms, making 22 classes of mutually equivalent formulas in all. The possibility of replacing the axiom $(a\to b)\to((b\to c)\to(a\to c))$ by a corresponding rule is considered. It is shown that the system of pure implication so altered is weakened, and that when conjunction is added to a certain such weakened system, the conservative property is lost. The equivalence of the system with alternation and an implicative (i.e., relatively pseudo-complemented) lattice is shown, and the author's paper of 1919 [Skr. Vid.-selsk. Kristiana. I. Mat. Nat. Kl. 1919, no. 3] cited as the source of this idea. Negation is defined in terms of a fixed Λ such that $\Lambda\to p$ is an axiom; it is shown that every property of negation is obtained from a suitable positive formula by specializing some variable to be Λ, and all formulas involving negation in this paper are actually derived that way. Under the heading "Generalization of the Glivenko theorem" is given what is essentially a more detailed version of Gödel's proof [Ergebn. Math. Kolloqu. **4**, 34–38 (1933)] that classical systems can be embedded in intuitionistic ones, with emphasis here on the predicate calculus rather than on number theory. In the following points some criticism is justified. In the preliminary account of formal systems, Skolem declares for a syntactical view of formalism, yet there is no clear distinction between use and mention; in particular it is not stated how parentheses, which are not listed in the alphabet, are to be regarded. The author's actual procedure would be correct from the standpoint of abstract (non-syntactical) formalisms, but this does not agree with the preliminary explanations. Again there is confusion in the use of letters 'S' and 'Σ' to denote systems, and the subscripts used in connection with these are not always consistent. There is no mention of the deduction theorem, and no use of the methods of Gentzen, although these methods could establish many of the results more convincingly than those of the author. The minimal calculus is not mentioned; the property $\neg\neg(a\vee\neg a)$ is derived from

$$(b\to a)\to((a\vee(a\to b))\to b)\to b)$$

whereas the Gentzen methods would show quickly that the premise $b\to a$ is superfluous. *H. B. Curry.*

Menger, Karl. The ideas of variable and function. Proc. Nat. Acad. Sci. U. S. A. **39**, 956–961 (1953).

The author distinguishes three distinct senses of the word function, viz., the logical, the scientific, and the mathematical. He argues that confusion between these notions accounts for the obscurity of many introductory treatments of analysis. The paper is suggestive, but the excessive condensation makes it difficult to understand the author's arguments in detail. *H. B. Curry* (State College, Pa.).

Hermes, Hans. Über den Begriff der Grenze in der Mathematik. Studium Generale **5**, 585–591 (1952).

Iséki, Kiyoshi. On the integers by a constructive method. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **2**, 93–104 (1952).

The integers, positive, negative, and zero, are constructed as classes of equivalent ordered pairs (x, y) of natural numbers. Two ordered pairs are equivalent if they have the same difference. The usual operations and relations are defined, and are proved to have the usual properties. In particular it is shown that the integers, so defined, form an integral domain. *O. Frink* (State College, Pa.).

Stipanić, Ernest. Le principe de la permanence de Hankel à la lumière de l'évolution historique de la notion du nombre depuis le nombre naturel jusqu'au nombre imaginaire. (Une contribution à la dialectique concrète du nombre.) Bull. Soc. Math. Phys. Serbie **5**, no. 1–2, 57–66 (1953).

***Schmidt, Arnold. Zum Verhältnis von Existenz und Widerspruchsfreiheit.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 205–207. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

***Sanchez-Mazas, Miguel. Mathematics as a science of formal structures (dialectics) and as a science of real structures (natural science).** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 218–223. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953. (Spanish)

*Greenwood, Thomas. **Valeur explicative des mathématiques.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 145–153. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

*Callahan, J. J. **Symbolism in mathematics and logic.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 166–171. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

*Dingler, Hugo. **Was ist konventionalismus?** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 199–204. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

*Rostand, François. **Quelques aspects de la psychologie des mathématiques.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 215–217. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

ALGEBRA

Majumdar, Kulendra N. On some theorems in combinatorics relating to incomplete block designs. Ann. Math. Statistics **24**, 377–389 (1953).

The author first proves: If in a balanced incomplete block design (b.i.b.) every pair of varieties occurs in λ blocks and any two blocks have λ' varieties in common, then the design is symmetrical. He then investigates conditions under which certain regularities in the block intersections arise. A typical result is the following: The necessary and sufficient condition for the existence in a b.i.b. of two blocks such that the number of varieties common to any other block and these two are equal is that these two blocks have $k+\lambda-r$ varieties in common. A b.i.b. is called resolvable (r.b.i.b.) if the blocks can be arranged in r groups each containing a complete replication. The design is called affine resolvable (a.r.b.i.b.) if $b=v+r-1$; the author proves that in an a.r.b.i.b. k or v is a perfect square according as r is odd or even. Furthermore, if $r\equiv 2, 3(4)$, then the square-free part of v/k cannot be divisible by a prime of the form $4n+3$. The author gives several parameter combinations for which by his theorem no design exists. *H. B. Mann.*

Slepian, David. On the number of symmetry types of Boolean functions of n variables. Canadian J. Math. **5**, 185–193 (1953).

*Singer, Theodore. **The theory of counting techniques.** Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 287–291. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

*Ashenhurst, Robert L. **The application of counting techniques.** Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 293–305. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

Consider $f(x_1, \cdots, x_n)$ which takes the value 0 or 1 where the x_j, $1\leqq j\leqq n$, take the value 0 or 1. There are $\mu=2^{2^n}$ such functions. If one function can be obtained from another by permuting the variables or "complementing" some of them then the functions are said to be of the same type. The number N_n of types of such functions of n variables is of interest to designers of electronic switching circuits [Synthesis of electronic computing and control circuits, by the Staff of the Computation Laboratory, Harvard Univ. Press, 1951; these Rev. **13**, 497] since it indicates the number of distinct pieces of hardware needed to generate all single output electronic switches which have n inputs.

Each of the μ functions above can be written as a sum of simple functions p_i, $i=2^n$, [loc. cit., p. 19] where the simple functions consist of a product of n terms the ith term being either x_j or $x_j^1=1-x_j=$complement of x_j. The finite group, O_n, generated by the permutations and "complementations" of the n variables, is represented in a natural way by permutation of the p_i. Let a class C be the collection of elements of O_n which leave each type invariant. Then $N_n=(1/2^n n!)\sum n_C 2^{k(C)}$, where n_C is the number of elements of C and $k(C)$ is the number of cycles in which the p_i are permuted by any element of class C of O_n. Furthermore there is a technique, too complicated to describe here, for computing N_n. Both the above are in effect given by Pólya [J. Symbolic Logic **5**, 98–103 (1940); these Rev. **2**, 65] where N_n is computed for $n=1, 2, 3, 4$. In the first paper this information is deduced in a relatively sophisticated way and N_n is computed for $n=1, 2, \cdots, 5, 6$. In the second and third papers the authors, referring to what must be an earlier version of the first paper [D. Slepian, Detailed enumeration of the number of Boolean (or switching) functions of n variables, unpublished memorandum on file at Bell Telephone Laboratories] (not available to reviewer) give a more elementary discussion of the computational technique, its derivation, and make some remarks about related problems in electronic switching circuits.

S. Sherman (Sherman Oaks, Calif.).

Jackson, James R. A partial ordering defined by certain matrices. Proc. Amer. Math. Soc. **4**, 429–430 (1953).

If for some column of a real matrix P the element in the mth row is negative and that in the m'th row is positive, write $m \dashv\!* m'$ and construct the transitive relation $\dashv$: $m \dashv m'$ if $m_i \dashv\!* m_{i+1}$ for some finite sequence of row indices with $m_1=m$ and $m_k=m'$. If never $m \dashv m$, P is called non-circulatory and this is proved equivalent to the possibility of permuting rows and (independently) columns to get $(q_{\mu\nu})$ with $q_{\mu\nu}>0$ implying $q_{mn}\geqq 0$ for all $m\leqq\mu$, $n\leqq\nu$. The definition is suggested by a mathematical model of industrial production. *W. Givens* (New York, N. Y.).

Medlin, Gene W. On bounds for the greatest characteristic root of a matrix with positive elements. Proc. Amer. Math. Soc. **4**, 769–771 (1953).

The author improves a theorem by A. Brauer [Duke Math. J. **19**, 553–562 (1952), theorem 39; these Rev. **14**, 836] on bounds for the greatest characteristic root of a matrix with positive elements He uses Ostrowski's [J. London Math. Soc. **27**, 253–256 (1952); these Rev. **14**, 126] recently found bound for the smallest component of the characteristic vector (normalized so that its largest component is unity) which corresponds to the greatest characteristic root of a positive matrix. *O. Taussky-Todd.*

Medlin, G. W. A note on a theorem of Parker. Amer. Math. Monthly **60**, 404–406 (1953).

Given matrices A, B and D with $DA=kA$, D and AB square, then AB has characteristic equation $x^{n-r}\phi(x)=0$

with degree $\varphi=\text{rank } A$ and $AB+D$ has characteristic equation $g(x)\phi(x-k)=0$. A term is omitted in equation (5). *W. Givens* (New York, N. Y.).

Mannos, Murray. Eigenvectors of matric polynomials. J. Research Nat. Bur. Standards **51**, 33–36 (1953).

The paper purports to prove as its main result: The matrices A and $f(A)$ have identical eigenvectors if and only if (1) $f'(\lambda)\neq 0$ for all eigenvalues λ of the matrix A corresponding to nonlinear elementary divisors, and (2) the values $f(\mu)$ are distinct for all eigenvalues μ of the matrix A corresponding to linear elementary divisors.

The theorem is false. A counter example is

$$A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad f(\lambda)=\lambda(\lambda-1).$$

While the theorem is easily corrected by omitting the last five words, the result remains trivial. *W. Givens.*

Olkin, Ingram. Note on 'The Jacobians of certain matrix transformations useful in multivariate analysis'. Biometrika **40**, 43–46 (1953).

The Jacobians of certain matrix transformations are evaluated by the techniques developed in a previous paper [Deemer and Olkin, Biometrika **38**, 345–367 (1951); these Rev. **13**, 855]. To quote the author's conclusion, Theorem 2 is concerned with an alternative form of the Toeplitz factorization which is a rotation of the rectangular coordinates. In Theorem 3, the relation between the correlation and covariance matrices is stated. In Theorem 4, a set of normalized rectangular co-ordinates is introduced, and in Theorem 5, these are related to the correlation matrix. In Theorems 6–8, respectively, the Cayley transformation for a general matrix, a symmetric and a triangular matrix is given. In Theorem 10, the Smith canonical form which yields the characteristic roots of a matrix is considered.

H. Chernoff (Stanford, Calif.).

Gautschi, Werner. The asymptotic behaviour of powers of matrices. Duke Math. J. **20**, 127–140 (1953).

For the $n\times n$ matrix $A=(a_{ij})$ with complex elements, define its norm (or absolute value) by

$$N(A)=\left(\sum_{i,j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}},$$

and let $\lambda_1,\cdots,\lambda_n$ be the eigenvalues of A. Assuming that A is not nilpotent (a natural restriction), the author proves that

$$1\leqq\frac{N(A^p)}{\left(\sum_{i=1}^{n}|\lambda_i|^2\right)^{\frac{1}{2}}}\leqq c\cdot p^{k-1}$$

for all positive integers p, where c is a positive constant depending only on A, and where k is the maximum multiplicity of any non-zero λ_i. The left-hand inequality follows readily from the unitary reduction of A to triangular form, while the proof of the right-hand inequality depends on the expression of A^p as a linear combination of matrices which depend on A but not on p, with coefficients given as functions of p. These inequalities are shown to be best possible.

The author next considers certain types of triangular matrices D (with zeros below the main diagonal), and obtains upper bounds for the individual elements of D^p; these bounds depend on the row in which the element occurs. As an example of results of this type, if $D=(d_{ij})$ is triangular, and $|d_{11}|>|d_{22}|>\cdots>|d_{nn}|>0$, it is proved that the modulus of any element in the ith row of D^p is $O(|d_{ii}|^p)$.

Next, setting $F_p(A)=(N(A^p))^{1/p}$ and $\omega_A=\max_i|\lambda_i|$, it is shown that $\omega_A\leqq F_p(A)$ for all positive integers p, and that $\lim_{p\to\infty}F_p(A)=\omega_A$; this improves a theorem due to Farnell [Amer. Math. Monthly **52**, 488–493 (1945); these Rev. **7**, 108]. The author also shows that $F_p(A)$ decreases monotonically toward ω_A as p runs through a strictly increasing sequence $\{p_r\}$ each term of which divides its successor. Similar results are obtained when $F_p(A)$ is replaced by $G_p(A)$, defined by

$$G_p(A)=\left(\max_i\sum_{j=1}^{n}|a_{ij}^{(p)}|\right)^{1/p},$$

where $A^p=(a_{ij}^{(p)})$. These results strengthen theorems due to Farnell [loc. cit.] and A. Brauer [Duke Math. J. **13**, 387–395 (1946); these Rev. **8**, 192]. A useful bibliography is given at the end of the paper. *I. Reiner.*

Motzkin, T. S., and Taussky, Olga. Pairs of matrices with property L. Proc. Nat. Acad. Sci. U. S. A. **39**, 961–963 (1953).

The previously studied property L of a pair of matrices A, B [cf. Trans. Amer. Math. Soc. **73**, 108–114 (1952); Nederl. Akad. Wetensch. Proc. Ser. A. **55**, 511–512 (1952); these Rev. **14**, 236, 723] is now characterized by the requirement that the curve in the projective (λ,μ,ν)-plane with equation $f(\lambda,\mu,\nu)\equiv|\lambda A+\mu B-\nu I|=0$ shall split into n lines. If 1) the discriminant of $f(\lambda,1,\nu)$ is $\equiv 0$ or all its roots have even order and 2) no $\lambda A+\mu B$ $(\lambda,\mu\neq 0,0)$ has a characteristic root of multiplicity $\geqq 3$ nor two different double characteristic roots, then A, B have property L. If all $\lambda A+\mu B$ are diagonable, A and B commute. Results are stated for an algebraically closed field of characteristic zero and detailed proofs and extensions to finite characteristic are to appear elsewhere. *W. Givens.*

Albert, A. A. Rational normal matrices satisfying the incidence equation. Proc. Amer. Math. Soc. **4**, 554–559 (1953).

The incidence matrix A of a finite projective geometry with $m+1$ points on a line is an n by n matrix $(n=m^2+m+1)$ satisfying $AA^T=A^TA=B=mI+N$ where N is the matrix of all 1's. It has been shown by Bruck and Ryser [Canadian J. Math. **1**, 88–93 (1949); these Rev. **10**, 319] that there exists a rational matrix X satisfying $X^TX=B$ if and only if (1) $m\equiv 0,3\pmod 4$ or (2) $m\equiv 1,2\pmod 4$ and $m=a^2+b^2$. In this paper it is shown by a constructive method that if $m=a^2+b^2$ there exists a normal rational matrix Y satisfying not only $Y^TY=B$ but also $YY^T=B$. *Marshall Hall, Jr.*

Reichel, Georg. Zur Transformationstheorie der Matrizen über dem Ring der ganzen p-adischen Zahlen. Math. Z. **57**, 75–85 (1952).

Let $\mathfrak{R}_p$ be the ring of p-adic integers, and let A be a non-singular matrix over $\mathfrak{R}_p$. This paper deals with the problem of finding a canonical form for A under similarity transformations $A\to TAT^{-1}$, where T also has elements in $\mathfrak{R}_p$. Canonical forms are obtained which depend on the elementary divisors (mod p) of $A-xI$, and a partial solution is given to the problem of deciding whether two such forms are similar. This problem is solved completely for the case where $A-xI$ has exactly one elementary divisor (mod p), and also for the case where A is a 2×2 matrix. As a corollary to this latter result, the author determines the class number

of the automorphism group of the abelian group of type (p^{n+1}, p^{n+1}). The reviewer wishes to point out that in the statement of Theorem 1 of this paper, the congruence signs should be replaced by ordinary equality signs.

I. Reiner (Urbana, Ill.).

Makar, Raouf H. The binomial series in infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 408–414 (1952).

If A is an infinite self-associative matrix satisfying a polynomial equation with complex number coefficients and r is a positive integer, the series $\frac{1}{r}\sum_{s=0}^{\infty}\binom{r+s-1}{s}A^s$ converges if and only if its general term converges to the zero matrix as $s\to\infty$. Then its sum is the rth power of the unique two sided reciprocal of $I-A$. *W. Givens.*

Bott, R., and Duffin, R. J. On the algebra of networks. Trans. Amer. Math. Soc. **74**, 99–109 (1953).

Inspired by electrical and mechanical applications the authors consider a quadratic form $E=\sum_1^n\sum_1^n G_{ij}v_iv_j$, where E is related to energy and the coordinates v_i represent displacement. The case of interest is where the coordinates are subject to $n-m$ homogeneous linear constraints. Let V be the subspace to which the coordinates are constrained. The discriminant D is the determinant of G relative to V. It is a homogeneous multilinear function of G_{ij}. The constrained inverse T of G is an $n\times n$ matrix such that the range of T is in V and TG is the identity in V. As in the case of an ordinary inverse, the constrained inverse of G exists if and only if $D\neq 0$.

When G is diagonal, let the diagonal elements be $g_1, \cdots, g_n$. Let $\psi=\log D$, $\psi_i=\partial\psi/\partial g_i$ and $\psi_{ij}=\partial^2\psi/\partial g_i\partial g_j$. Then $-\psi_{ij}$ is the square of a rational function. Conversely, any homogeneous multilinear function which satisfies this perfect square condition is the discriminant of a diagonal matrix. These results are related to the work of Cohn [Proc. Amer. Math. Soc. **1**, 316–324 (1950); these Rev. **12**, 148]. All of the derivatives of ψ are expressible in terms of the matrix elements of T. Thus $\psi_i=T_{ii}$ and $\psi_{ij}=-(T_{ij})^2$.

S. Sherman (Sherman Oaks, Calif.).

Wall, G. E. The Galois group of a binomial polynomial. Proc. London Math. Soc. (3) **3**, 195–210 (1953).

The question of determining the Galois group of the binomial polynomial x^m-a, where a is a given number in a certain ground field R, has often been discussed in the past. Some of the results are well known, but even in the case of the rational field the situation may be quite complicated. Rather surprisingly, the problem does not appear to have been treated in its full generality before.

The author observes that the original question may be regarded as equivalent to the following two problems. (1) the determination of all possible Galois groups of binomial polynomials. This is essentially equivalent to finding all subgroups of the linear permutation group L_m consisting of the transformations $x'=sx+t$, where t runs over all residue classes mod m and s over those prime to m. (2) the solution of a set of equations expressing the condition that the Galois group of x^m-a is a subgroup of a given possible Galois group, i.e., a suitable subgroup of L_m. Both these problems are completely solved in the present paper subject only to the (obvious) restriction that the characteristic of R should be prime to m. The first problem possesses independent interest, as it involves a very thorough analysis of the group L_m.

The author has asked the reviewer to draw attention to a possible source of confusion arising out of the notation introduced in §3, p. 197. The letter s should, in principle, be interpreted as the automorphism of $R(\epsilon)$ over R induced by S. It is only when s is applied to ϵ itself that ϵ^s becomes an actual power, and s may be considered an element of $\mathfrak{R}$.

W. Ledermann (Manchester).

Abstract Algebra

Szász, G. Die Unabhängigkeit der Assoziativitätsbedingungen. Acta Sci. Math. Szeged **15**, 20–28 (1953).

Let S be any set containing ν elements where ν may be finite or infinite. The author shows that if $\nu>3$, a multiplication may be defined in S in such a way that the associative law holds for all triples of elements of S except one arbitrarily designated triple. The set of all associativity conditions therefore constitutes an independent set of axioms. If $\nu\leqq 3$ this is not so, and independent sets of associativity conditions are given for the cases $\nu=2$ and $\nu=3$.

D. C. Murdoch (Vancouver, B. C.).

Klein-Barmen, Fritz. Zur Theorie der Operative und Assoziative. Math. Ann. **126**, 23–30 (1953).

By "Operativ" the author means a set M closed under a single-valued binary operation $\circ$. An "Assoziativ" is an Operativ $M(\circ)$ such that $a\circ(b\circ c)=(a\circ b)\circ c$ for all $a, b, c\in M$. The paper is chiefly concerned with elementary properties of elements of finite order in an Assoziativ.

A. H. Clifford (Baltimore, Md.).

Klein-Barmen, Fritz. Pseudoverband und Flechtwerk. Math. Ann. **126**, 138–143 (1953).

The concept of "pseudo-lattice" M, due to D. Ellis [Publ. Math. Debrecen **1**, 205–208 (1950); these Rev. **12**, 472] is specialized to the case of a "Flechtwerk" F, by assuming that M is a group under $\cap$ with identity e. It is shown, for example, that if M is finite with elements $a_1, \cdots, a_m=e$, and $m\geqq 3$, then $a_1\cup\cdots\cup a_{m-1}=e$.

G. Birkhoff (Cambridge, Mass.).

McLaughlin, J. E. Structured theorems for relatively complemented lattices. Pacific J. Math. **3**, 197–208 (1953).

The author considers a relatively complemented lattice L of finite dimension n. Among his results: L is isomorphic to a direct union $a/z\vee b/z$ if and only if each point of L is a point of exactly one of a, b and, in addition, $\rho(a)+\rho(b)\leqq n$; if L is simple then for each point p and non-negative integer $k<[(n+1)/2]$ the dimension of a_p^k/z is at least $2k+1$. In a previous paper [Duke Math. J. **18**, 73–84 (1951); these Rev. **12**, 667] the author has given an upper bound (depending on the dimension n) for the number of transposes required to establish projectivities between points; results are now obtained for the structure of the lattices in which this upper bound is attained.

I. Halperin (Kingston, Ont.).

Ikushima, Isaku. On π-regular ring. J. Osaka Inst. Sci. Tech. Part I. **2**, 91–95 (1950).

A ring R is said to be π-regular if for every $a\in R$ there exists an integer n and an $x\in R$ so that $a^nxa^n=a^n$. Let R_R denote the set of power principal right ideals (i.e., the principal right ideals generated by the a^n, where $a^nxa^n=a^n$)

and L_R the set of power principal left ideals of R. Let $a^l=\{x \in R \mid xa=(0)\}$ where a is a right ideal. The author proves the following. 1) The mapping defined from R_R to L_R by $a \to a^l$ is a one-to-one map onto. It is inverse to the mapping from L_R to R_R defined in a symmetric way, and is anti-monotonic. 2) The center, Z, of R is π-regular and Z_R is a complemented Boolean algebra. Theorem 1) generalizes the analogous result of von Neumann for regular rings.

I. N. Herstein (Philadelphia, Pa.).

Villamayor, Orlando E. On the concept of filter in ring theory. Revista Unión Mat. Argentina **15**, 173–180 (1953).

Let R be a ring and B a left R-module which is an R-homomorphic image of R where R itself is viewed as an R-module. Let σ be this R-homomorphism. An element u of B is said to be an R-unit vector if $ru=\sigma(r)$ for every $r \in R$. A left filter is then defined to be the inverse image of an R-unit vector. The author then proceeds to prove some elementary, and fairly obvious, results about such filters.

I. N. Herstein (Philadelphia, Pa.).

Rabin, Michael. Sur la représentation des idéaux par des idéaux primaires. C. R. Acad. Sci. Paris **237**, 544–545 (1953).

The main result of this note is the following. In a commutative ring R a necessary and sufficient condition that every ideal be the intersection of finitely many strongly primary ideals is that for every ideal A there exists an integer $N(A)>0$ such that every strictly ascending sequence $A \subset A:B_1 \subset \cdots \subset A:(B_1B_2\cdots B_k)$ of quotients be of length $k<N(A)$.

E. R. Kolchin (New York, N. Y.).

Kasch, Friedrich. Invariante Untermoduln des Endomorphismenrings eines Vectorraums. Arch. Math. **4**, 182–190 (1953).

A theorem, usually referred to as the Brauer-Cartan-Hua theorem states that any subdivision ring of a division ring which is carried into itself by all inner automorphisms of the division ring is either contained in the center or is the full division ring. On the other hand, it is known that for finite normal extensions of a field, a normal basis exists. In this paper, the author finds extensions of both these results.

Let A be the ring of all linear transformations of a left vector space B over a division ring K, B of dimension $n>1$, but not necessarily finite. If $K \neq GF(2)$ or $n>2$ then any subring, R, of A which is carried into itself by all inner automorphisms must either be contained in the center of A or must be a dense ring of transformations on B; in particular, if A satisfies the descending chain condition then either R is contained in the center of A or $R=A$. He uses the methods of this proof to extend results of Hua about the submodule generated by the multiplicative commutators of A. Defining normal basis in a manner which extends the notion in the case of finite field extensions, the author proves that if n is finite and K possesses a normal basis over its center Z, then A also possesses a normal basis over Z. Using the cyclicity of central simple division algebras over number fields, the author shows that any central simple algebra (finite-dimensional) over a number field Z possesses a normal basis over Z.

I. N. Herstein (Philadelphia, Pa.).

Greger, Karl. Über einen Satz von Gauss. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **23**, no. 16, 2 pp. (1953).

Let R be any commutative ring with unit. Let $f(x)$, $g(x)$ be polynomials with coefficients in R. Let I, J, K denote the ideals generated by the coefficients of f, g and fg. The author gives a brief, elementary proof of the following theorem: if $I=J=R$, then $K=R$. He refers to a paper of Dedekind [Ges. Math. Werke, vol. II, Vieweg, Braunschweig, 1931, pp. 28–39] but does not explicitly note that Dedekind has the stronger result that $I^{n+1}J=I^nK$, where n is the degree of g.

I. Kaplansky (Chicago, Ill.).

Krull, Wolfgang. Über Polynomzerlegung mit endlich vielen Schritten. Math. Z. **59**, 57–60 (1953).

The author exhibits an infinite sequence $K_1, K_2, \cdots$ of distinct finitely generated well-defined extensions of a field K_0 such that for every irreducible polynomial $p(X) \in K_0[X]$ there exists a number n^*, which can be computed in a finite number of steps, with the property that $p(X)$ is irreducible over K_n for every $n>n^*$. In the author's sequence K_0 is the field of rational numbers and $K_1, K_2, \cdots$ are the imaginary quadratic fields; he asks if there are others. If K is a field such that either $K=K_0$ or else $K=K_n$ for some $n>0$, and if for each $n>0$ it can be decided in a finite number of steps whether $K=K_n$, then K need not be well-defined and one even need not be able to find out in a finite number of steps whether $K=K_0$; nevertheless the author shows that every polynomial over K_0 can be completely factored over K in a finite number of steps. This work has close connection with a paper by M. Kneser [Math. Z. **57**, 238–240 (1953); these Rev. **14**, 613].

E. R. Kolchin (New York, N. Y.).

Habicht, Walter. Über Polynomabbildungen. Math. Ann. **126**, 149–176 (1953).

The author is concerned with the following theorem. Let n be odd and let $f_1, \cdots, f_n$ be polynomials in $x_1, \cdots, x_n$ making $\sum x_i f_i$ identically 0. Then the polynomials have a common (non-trivial) zero. This theorem is known over the field of real numbers; indeed it is then a special case of the Poincaré-Brouwer theorem on the non-existence of tangent vector fields on even-dimensional spheres. The project is to prove it over any real closed field. The author observes that the theorem falls within the scope of the methods of Tarski [A decision method for elementary algebra and geometry, 2nd ed., Univ. of California Press, 1951; these Rev. **13**, 423]; in other words, it is of the right form so that its truth over any real closed field follows from its truth over the real field. Nevertheless the author feels that this does not abolish the need for a purely algebraic proof. Conceivably this might be done by the methods of elimination theory or of algebraic geometry. But the author instead carries through the program of generalizing to an arbitrary real closed field enough combinatorial topology to make the usual proof work.

I. Kaplansky (Chicago, Ill.).

Krasner, Marc. La non-existence des extensions d'une certaine forme. C. R. Acad. Sci. Paris **237**, 370–372 (1953).

A recent result of the author [Amer. J. Math. **75**, 112–116 (1953); these Rev. **14**, 613] is generalized as follows. Let k be a field which is not an algebraic extension of a finite field, and K be a separable algebraic extension of k. Let $a(x) \in k[x]$, and x^m be the highest power of x dividing $a(x)$. Suppose that every $\alpha \in K$ is a zero of a polynomial $f_\alpha(x) \in k[x]$ of the form $f_\alpha(x)=x^{n(\alpha)}+a(x)+\phi(x;\alpha)$, where $n(\alpha)>m$ and where $\phi(x;\alpha) \in k[x]$ is a polynomial in x of degree $<m$, $n(\alpha)$ and $\phi(x^p \alpha)$ depending on α. Then $(K:k) \leqq m$. That the proof of this theorem is incomplete is pointed out in the note reviewed below, where the error is rectified. For another generalization of the author's original theorem, see Naka-

yama [Canadian J. Math. 5, 242–244 (1953); these Rev. 14, 719]. *R. D. Schafer* (Storrs, Conn.).

Krasner, Marc. Compléments à ma note précédente "Sur la non-existence des extensions d'une certaine forme". C. R. Acad. Sci. Paris **237**, 685–687 (1953).

In this note the author closes a gap in the proof of the theorem above. He also proves the following generalization of this theorem which is related to the result of Nakayama mentioned above. Let K be a separable algebraic extension of a field k, which is not an algebraic extension of a finite field. Let p be a positive integer, and D be a fixed finite set of nonzero elements of K. If every $\alpha \in K$ is a zero of a polynomial $f_\alpha(x) \in k[x]$, which is of the form

$$f_\alpha(x) = \sum_{j=0}^{p(\alpha)} d_j(\alpha) x^{n_j(\alpha)} + \phi(x;\alpha),$$

where $0 < p(\alpha) \leqq p$, $m = n_0(\alpha) < n_1(\alpha) \leqq \cdots \leqq n_{p(\alpha)}(\alpha)$, and where $\phi(x;\alpha)$ is of degree $< m$ and the $d_j(\alpha)$ belong to D, then $(K:k) \leqq m$. *R. D. Schafer* (Storrs, Conn.).

Krull, Wolfgang. Über eine Verallgemeinerung des Normalkörperbegriffs. J. Reine Angew. Math. **191**, 54–63 (1953).

The author reviews a definition and some results of D. Barbilian [Solutia exhaustiva a problemai lui Steinitz, Acad. Repub. Pop. Române. Stud. Cerc. Mat. **2**, 189–253 (1950), unavailable in this country], providing proofs which are said to be simpler, and further results. Let L be an extension of a field K. Then L is called normal over K if for every intermediate field M the relative algebraic closure M^* of M in L is normal (in the usual sense) over M. If L has the property that every M is uniquely determined by the automorphism group $U(M)$ of L over M, then L is normal over K and, if the characteristic $p = 0$, conversely; if $p > 0$ the converse fails but a certain weaker conclusion is obtained. Various further results are found, and constructive aspects of normal extensions are explored. Some open questions are discussed, the most important one being: Do there exist transcendental normal extensions which are not algebraically closed? *E. R. Kolchin* (New York, N. Y.).

Wolf, Paul. Galoissche Algebren mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers. I. Math. Nachr. **9**, 281–300 (1953).

In a previous paper [Math. Nachr. **9**, 201–216 (1953); these Rev. **15**, 6] the author gave a new treatment and an extension of Hasse's theory of Galois algebras [cf. H. Hasse, J. Reine Angew. Math. **187**, 14–43 (1949); these Rev. **11**, 576]. The generalized Galois algebras $\mathfrak{K}$ with Galois group $\mathfrak{G}$ over a field Ω were characterized by classes $\mathfrak{C}$ of associated normal factors C which were elements of the direct product $G \times G$ over Ω of the group algebra G of $\mathfrak{G}$ with Ω as ground field. In the present paper, the following question is treated: Suppose that Ω is normal of finite degree over a subfield Ω_0. What are the necessary and sufficient conditions on $\mathfrak{C}$ in order that $\mathfrak{K}$ considered as algebra over Ω_0 possesses a group $\mathfrak{G}_0$ of automorphisms which contains $\mathfrak{G}$ as normal subgroup and is such that $\mathfrak{G}_0/\mathfrak{G}$ induces in Ω the Galois group $\mathfrak{g}$ of Ω/Ω_0; $\mathfrak{G}_0/\mathfrak{G} \simeq \mathfrak{g}$? This is a generalization of a problem treated by Hasse [Math. Nachr. **1**, 40–61, 213–217, 277–283 (1948); Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1947**, no. 8 (1949); these Rev. **10**, 426, 503; **11**, 155] who considered the case of an abelian group $\mathfrak{G}$. The results of the author for the general case are quite similar to those of Hasse. If $\mathfrak{K}$ possesses a group $\mathfrak{G}_0$ of automorphisms of the required type, a system $\mathfrak{B} = (\cdots, B_s, \cdots)$ of elements of the group ring G of $\mathfrak{G}$ can be defined where s ranges over the elements of $\mathfrak{g}$. Actually, certain "associated" systems are to be considered as equivalent. These B_s have to satisfy two sets of relations. In the first set, the invariants of the group extension $\mathfrak{G}_0$ of $\mathfrak{G}$ by means of $\mathfrak{g}$ enter, while in the second system the normal factor C belonging to $\mathfrak{K}$ appears too. The main result is that these conditions are also sufficient: If, for given C and for an abstractly given group extension of $\mathfrak{G}$ by means of $\mathfrak{g}$, a system $(\cdots, B_s, \cdots)$ exists which satisfies these conditions, then the problem formulated above has a solution. In the general case, this does not necessarily mean that $\mathfrak{K}$ is a generalized Galois algebra over Ω_0 since for this, the additional condition has to be satisfied that $\mathfrak{K}$ as Galois module over Ω_0 is isomorphic to the group ring of $\mathfrak{G}_0$. However, if $\mathfrak{K}$ is a Galois algebra (i.e., a commutative and semisimple generalized Galois algebra) over Ω, the same is true for $\mathfrak{K}$ over Ω_0. Thus, the results of the author give a necessary and sufficient condition that a Galois algebra $\mathfrak{K}/\Omega$ with Galois group $\mathfrak{G}$ may be considered as a Galois algebra $\mathfrak{K}/\Omega_0$ with a Galois group isomorphic to a given extension of $\mathfrak{G}$ by means of the Galois group $\mathfrak{g}$ of Ω/Ω_0.

In the rest of the paper, various interpretations are given for the relations between the B_s, and the existence of solutions and the construction of solutions are discussed. In the case of an abelian group $\mathfrak{G}$, Hasse made the conjecture that the existence of a system $(\cdots, B_s, \cdots)$ satisfying the second system of relations implies that, for a suitable choice of the normal factor C of the Galois algebra $\mathfrak{K}$, the first system of relations holds too. The author extends this conjecture to the present case and states it in several forms. However, the question remains open whether or not the conjecture is true. *R. Brauer* (Cambridge, Mass.).

Okugawa, Kôtaro. Extensions of the ground field in the theory of algebraic differential equations. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 257–265 (1953).

Let $\mathfrak{F}$ be a differential field (ordinary or partial) of characteristic 0, $\mathfrak{G}$ an extension of $\mathfrak{F}$, and $\mathfrak{p}$ a prime differential ideal in the differential ring of differential polynomials over $\mathfrak{F}$ in a finite number of differential indeterminates. The author proves the following theorem which extends a result of Ritt for ordinary differential fields [Differential algebra, Amer. Math. Soc. Colloq. Publ., v. 33, New York, 1950, see p. 51; these Rev. **12**, 7]: $\mathfrak{G}\mathfrak{p}$ is a perfect differential ideal; the dimensions of the essential prime divisors $\mathfrak{P}_1, \cdots, \mathfrak{P}_t$ of $\mathfrak{G}\mathfrak{p}$ are all equal to the dimension of $\mathfrak{p}$; every parametric set for $\mathfrak{p}$ is one for every $\mathfrak{P}_j$, and the characteristic set of $\mathfrak{P}_j$ with respect to such a parametric set has the same leaders as that of $\mathfrak{p}$. He then introduces the concept of differential independence (which with equal justification might have been called differentially algebraic disjointness) of two extensions of $\mathfrak{F}$, and proves a succession of propositions analogous to results on independence (i.e., algebraic disjointness) of field extensions in A. Weil's Foundations of algebraic geometry [Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; these Rev. **9**, 303]. *E. R. Kolchin* (New York, N. Y.).

Magnus, Wilhelm. Algebraic aspects in the theory of systems of linear differential equations. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **BR-3**, i+35 pp. (1953).

The author: (1) demonstrates a necessary and sufficient condition, due to Friedrichs [Comm. Pure Appl. Math. **6**, 1–72 (1953); these Rev. **15**, 80], for an element of the free

associative algebra with generators $x_1, \cdots, x_n$ over a field of characteristic 0 to be in the Lie algebra generated by $x_1, \cdots, x_n$, and derives therefrom (à la Friedrichs) the Baker-Hausdorff theorem concerning $\log e^x e^y$ and $e^{-z}ye^z$; (2) shows that if A, Y are functions (of a real variable t) belonging to a ring and if $dY/dt = AY$, $Y(0)=1$, then, under certain conditions, $Y=\exp \Omega$, where Ω is a series the terms of which can be found recursively and the formula for which is a continuous analog of the Baker-Hausdorff formula; (3) finds, when A, Y are square matrices, certain conditions under which the formula for Ω is valid and gives a solution of the differential equation. *E. R. Kolchin.*

Kowalsky, Hans-Joachim, und Dürbaum, Hansjürgen. Arithmetische Kennzeichnung von Körpertopologien. J. Reine Angew. Math. **191**, 135–152 (1953).

(1) A necessary and sufficient condition for a topological division ring to have a valuation onto an ordered abelian group is that the ring be of type V [terminology following Kaplansky, Duke Math. J. **14**, 527–541 (1947); these Rev. **9**, 172] and have bounded multiplicative commutator group. This same result was sketched recently by I. Fleischer [C. R. Acad. Sci. Paris **236**, 1320–1322 (1953); these Rev. **14**, 720]. There are also two unpublished proofs of the theorem in the commutative case: by R. Turyn [Harvard undergraduate thesis, 1951] and by I. Fleischer [doctoral dissertation, Univ. of Chicago, 1952]. The proofs exploit the fact that in a locally bounded topology on K a base at zero is $\{xU \mid 0 \neq x \in K\}$ where U is some "Fastordnung" in K; i.e., $U+U \subset zU$ for some z, $UU \subset U$, 0 and $1 \in U$, each $xU \subset Uy$ for some y, and each $Uy \subset zU$ for some z. "Type V" imposes enough further conditions on U to prove the result (1) as well as the following. (2) A topological division ring of type V with no topologically nilpotent elements has a valuation (onto a generally nonabelian ordered group) if and only if the ring contains an invariant, bounded neighborhood of zero. (3) A topological division ring of type V containing a topological nilpotent has a valuation if and only if every nilpotent times a neutral or nilpotent is nilpotent; in this case the value group may be taken as a group of real numbers. Examples show that the necessary and sufficient conditions in (1) and (2) are not automatically satisfied. *D. Zelinsky* (Evanston, Ill.).

Ikushima, Isaku. G-radical of topological rings. J. Osaka Inst. Sci. Tech. Part I. **2**, 81–84 (1950).

Definitions and theorems concerning the Brown-McCoy radical in G-rings that coincide to a large extent with those in the later paper by Iseki [Anais Acad. Brasil. Ci. **25**, 79–86 (1953); these Rev. **14**, 720]. There are also a few related results on bounded G-rings. *D. Zelinsky.*

Theory of Groups

Tvermoes, Helge. Über eine Verallgemeinerung des Gruppenbegriffs. Math. Scand. **1**, 18–30 (1953).

Following Dörnte [Math. Z. **29**, 1–19 (1928)] a set N of elements is called an n-group if, corresponding to any ordered set $a_1, a_2, \cdots, a_{n+1}$ of elements of N, there is a uniquely defined "product" in N designated by $a_1a_2\cdots a_{n+1}$. Moreover this composition obeys the associative law

$$(a_1\cdots a_{n+1})a_{n+2}\cdots a_{2n+1} = a_1\cdots a_i(a_{i+1}\cdots a_{i+n+1})a_{i+n+2}\cdots a_{2n+1}$$

for $i=1, 2, \cdots, n$, and the equations $a_1\cdots a_n x = a$ and $ya_2\cdots a_{n+1}=a$ are always soluable in N for x and y. A 1-group is clearly a group. An n-group can evidently be constructed from the elements of a group G by taking the product of $n+1$ elements in G to be the product of the corresponding elements of the n-group. It is shown that every n-group N can be embedded in a group G in such a way that the product of $n+1$ elements of N coincides with their product in G. Moreover, there exists exactly one such enveloping group G with the property that N is a primitive coset modulo a normal subgroup of G with cyclic factor group of order n. This is the main result of the paper although some further relationships between N and G are obtained. *D. C. Murdoch* (Vancouver, B. C.).

Sade, Albert. Contribution à la théorie des quasi-groupes: diviseurs singuliers. C. R. Acad. Sci. Paris **237**, 372–374 (1953).

On appelle diviseur normal d'un quasigroupe toute relation d'équivalence dans ce quasigroupe, qui est préservée par les multiplications et divisions à gauche et à droite. L'auteur généralise cette notion, en considérant les relations d'équivalence (qu'il appelle diviseurs quasi-normaux), pour lesquels cette préservation a lieu quand les éléments congrûs sont en dehors d'un certain sous-ensemble "singulier" du quasi-groupe, qui est lui-même une classe d'équivalence de la relation considérée. Le produit de deux classes "régulières" (c'est-à-dire distinctes de l'ensemble singulier) est ou bien une telle classe, ou bien la réunion d'une telle classe et de la classe singulière. Ceci permet d'organiser l'ensemble de ces classes régulières en un quasigroupe, et de définir la notion correspondante d'"homomorphisme singulier". Les diviseurs quasi-normaux peuvent se mettre sous une forme particulièrement simple à l'aide de certains autotopies. *M. Krasner* (Paris).

Sade, Albert. Contributions à la théorie des quasi-groupes: quasi-groupes obéissant à la "loi des keys" ou automorphes par certains groupes de permutations de leur support. C. R. Acad. Sci. Paris **237**, 420–422 (1953).

On dit qu'un quasigroupe obéit à la "loi des keys" si, pour tous ses éléments x, y, on a $(xy)y=x$; on dit qu'elle obéit à la loi des keys bilatère, si l'on a $(xy)y=y(yx)=x$. L'auteur étudie, dans les quasigroupes, satisfaisant à ces lois, la structure des autotopies, qui les préservent. D'autre part, il fait l'étude d'une classe particulière de quasigroupes, obéissant à la loi des keys bilatère. Finalement, il considère, sur l'anneau des restes des entiers (mod n), des lois de composition $a \times b$, et caractérise celles de ces lois, qui sont les lois de quasigroupes, préservés soit par le groupe linéaire de cet anneau, soit par son groupe des translations.
M. Krasner (Paris).

Clifford, A. H. A class of d-simple semigroups. Amer. J. Math. **75**, 547–556 (1953).

A semigroup S with an identity 1 is said to be d-simple if, given any two elements a and b of S, there exists an element p of S such that $aS=pS$ and $Sp=Sb$. The set of all elements x of S for which there exist elements y of S such that $xy=1$ is a semigroup and is called the right-unit subsemigroup P of S. The paper proves that if two idempotent elements of S always commute, then P has the following properties: P has an identity; the right-cancellation law holds; the intersection of two principal left-ideals is always a principal left-ideal. Moreover, if a semigroup Q has these three properties, then there is a d-simple semigroup, two idempotent

elements of which always commute, whose right-unit subsemigroup is isomorphic to Q. This is one step in the determination of all regular d-simple semigroups.

H. A. Thurston (Bristol).

***Stolt, Bengt. Über Axiomensysteme die eine abstrakte Gruppe bestimmen.** Thesis, University of Uppsala, 1953. Almqvist & Wiksells, Uppsala, 1953. 99 pp.

The relation $xy=z$ is generalized to a ternary relation $xy \supset z$, subject to an associativity condition. Axioms are considered asserting existence or uniqueness of product, left- and right-quotient; of idempotents, left- and right-identities, and left- and right-identities relative to a particular element; and of left- and right-inverses relative to these various sorts of identities. Among these axioms, sets are determined which characterize groups. *R. C. Lyndon.*

***Krause, Hans Ulrich. Gruppenstruktur und Gruppenbild.** Thesis, Eidgenössische Technische Hochschule, Zürich, 1953. 45 pp.

A graph G, assumed connected and of finite degree, belongs (as 'Gruppenbild') to the group $\mathfrak{G}$ if $\mathfrak{G}$ is a simply-transitive group of automorphisms of G. This is equivalent to Dehn's definition, where G is constructed with a vertex for each g in $\mathfrak{G}$, and a connecting edge whenever $g_i g_j^{-1}$ is in a prescribed set of generators. Finite groups have a common graph (e.g., simplex) if and only if they have the same order. The main result states that two finitely generated abelian groups have a common graph if and only if they have the same rank, and their torsion groups the same order.

The 'if' follows easily from a product construction. For free abelian groups, the proof may be completed by a study of the automorphisms of G leaving a given vertex fixed; alternatively, it is shown that the rank is $\lim (\ln V(r)/\ln r)$ where $V(r)$ is the number of vertices within distance r of a given vertex. The full theorem is proved by adjusting given f: $G \sim G'$ to induce a correspondence F: $\mathfrak{G}$ onto $\mathfrak{G}'/\mathfrak{T}'$ carrying 0 into 0. Then $\Phi(a) = F(a+b) - F(a)$ satisfies $\sum_{a'}[\Phi(a) - \Phi(a')] = 0$, summed over a' adjacent to a. Ordering $\mathfrak{G}'/\mathfrak{T}'$, the harmonic function Φ, which attains a maximum, must be constant and hence 0. Thus F is a homomorphism, and the theorem follows. *R. C. Lyndon.*

Michiura, Tadashi. Commutativity in simply ordered groups. J. Osaka Inst. Sci. Tech. Part I. **3**, 39–41 (1951).

The author gives the following theorem, which the reviewer believes to be incorrect. Let G be a simply ordered non-Archimedean group. Then G is commutative if any two non-Archimedean elements of G are permutable. A counterexample is provided by Example 7 on p. 216 of Garrett Birkhoff's "Lattice theory" [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. **10**, 673]. *A. H. Clifford* (Baltimore, Md.).

Neumann, B. H. A note on means in groups. J. London Math. Soc. **28**, 472–476 (1953).

W. R. Scott [Amer. J. Math. **74**, 667–675 (1952); these Rev. **13**, 910] finds necessary and sufficient conditions for the existence on a group of a set $\{f_n\}$ of Schimmack means [Math. Ann. **68**, 125–132 (1909)]. The author answers affirmatively the question of Scott as to whether there exists a non-abelian group with a set of Schimmack means. Let $[x, y]$ be the commutator $x^{-1}y^{-1}xy$. It is proved that a set of necessary and sufficient conditions for the existence of a set of Schimmack means on a group G is: (a) unique nth roots exist for each element of G and for each positive integer n; and (b) $[[x, y], x] = e$, the identity of G, for every pair of elements x and y of G. Condition (b) replaces a more complicated condition of Scott [loc. cit.]. For his non-abelian example, the author lets G be the cartesian product of three copies of the group of rationals with addition defined by $(r, s, t) + (r', s', t') = (r+r', s+s', t+t'-r's)$. The Schimmack means on G turn out to be closely related to the arithmetic means of the components and their cross-products.

F. Haimo (St. Louis, Mo.).

Kertész, A. On a theorem of Kulikov and Dieudonné. Acta Sci. Math. Szeged **15**, 61–69 (1953).

In this paper the author extends a previous result of his [Acta Math. Acad. Sci. Hungar. **3**, 225–232 (1952); these Rev. **14**, 945] in the same way as Dieudonné's theorem [Portugaliae Math. **11**, 1–5 (1952); these Rev. **13**, 720] extended Kulikoff's [Mat. Sbornik N.S. **16(58)**, 129–162 (1945); these Rev. **8**, 252]. The new theorem reads as follows. A primary abelian group G is a direct sum of groups of rank one if and only if: (1) every element of infinite height in G can be written as p times another element of infinite height, (2) there exists a subgroup B such that B possesses a principal system relative to G, i.e., a maximal independent subset of B such that no element can be exchanged for one of greater height without violating independence, (3) G/B is a direct sum of cyclic groups.

I. Kaplansky (Chicago, Ill.).

Berman, S. D. On certain properties of integral group rings. Doklady Akad. Nauk SSSR (N.S.) **91**, 7–9 (1953). (Russian)

The author studies the group algebra R, over the ring of rational integers, of a group G of finite order n. An element $u \in R$ is called normal if $uu^* = u^*u$ ($u \to u^*$ being the canonical involution of R). An element $u \in R$ such that $u^t = \pm 1$ for some rational integer $t > 0$ is called a root of unity, and the smallest such t is called the order of u; the elements of $\pm G$ are obviously normal roots of unity. Theorem 1. Every normal root of unity in R belongs to $\pm G$. Theorem 2. The order of a root of unity in R divides n. Theorem 3. R fails to contain a nilpotent element if and only if G is abelian or is hamiltonian of order $2^m t$, where 2 belongs to an odd exponent modulo t. Theorem 4. The following three conditions are equivalent. I. Every root of unity of R belongs to $\pm G$. II. Every element of R is normal. III. G is abelian, or is hamiltonian of order power of 2.

E. R. Kolchin (New York, N. Y.).

Berman, S. D. On isomorphism of the centers of group rings of p-groups. Doklady Akad. Nauk SSSR (N.S.) **91**, 185–187 (1953). (Russian)

Let G be a group of finite order, K a field, $R(G, K)$ the group algebra of G over K. Letting n be the smallest integer > 0 such that $x^n = 1$ for every x in G, and letting ϵ be a primitive nth root of unity, the author calls elements x, y of G K-conjugate if x is conjugate to a power y^m such that $\varphi(\epsilon) = \epsilon^m$ for some automorphism φ of $K(\epsilon)$ over K. The group G splits up into K-conjugate classes; each K-conjugate class of G itself splits up into conjugate classes of G. Theorem. Let p be a prime number $\neq 2$, and let G, H be p-groups. $R(G, K)$ and $R(H, K)$ have isomorphic centers if and only if there is a one-to-one correspondence between the set of K-conjugate classes of G and that of H, such that corresponding K-conjugate classes contain the same number of conjugate classes. *E. R. Kolchin* (New York, N. Y.).

Maranda, Jean-Marie. On $\mathfrak{P}$-adic integral representations of finite groups. Canadian J. Math. **5**, 344–355 (1953).

Let $\mathfrak{H}$ be the group ring of a finite group $\mathfrak{G}$, of order N, over a ring $\mathfrak{O}$ of $\mathfrak{P}$-adic integers. Let $\mathfrak{P}^{k_0} \| N$. It is proved firstly, by means of the Hurwitz-Schur summation method, that for $k > k_0$ two (matric) representations of $\mathfrak{H}$ in $\mathfrak{O}$ are unimodularly equivalent if and only if they are so modulo $\mathfrak{P}^k$. When $\mathfrak{O}$ is complete a reduction, resp. decomposition, of a representation $\mathfrak{D}$ of $\mathfrak{H}$ in $\mathfrak{O}$ follows from one modulo $\mathfrak{P}^k$ provided $k > 2k_0$; the components are taken to be congruent to the given modular ones modulo $\mathfrak{P}^{k-k_0}$. The irreducible constituents for complete reductions of $\mathfrak{D}$ are unique up to order and unimodular equivalence if and only if such is the case for those modulo $\mathfrak{P}^k$ ($k > 2k_0$), and similarly for complete decompositions. *T. Nakayama* (Nagoya).

Takahashi, Shuichi. Cohomology groups of finite abelian groups. Tôhoku Math. J. (2) **4**, 294–302 (1952).

A concise algorithm for constructing the incidence matrices for a direct product of groups from those for the components is applied to finite abelian groups. If each cyclic component A_i has a single cell $(i)^p$ in each dimension p, the product cells $(m)^{p_m} \cdots (1)^{p_1}$ correspond to non-increasing sequences of integers, and a compact definition of the operator δ may be given. Applications: integral cohomology groups of finite abelian groups; existence of algebraic number fields with cohomology groups unbounded with increasing dimension. *R. C. Lyndon* (Ann Arbor, Mich.).

Kneser, Martin, und Puppe, Dieter. Quadratische Formen und Verschlingungsinvarianten von Knoten. Math. Z. **58**, 376–384 (1953).

By a V-group is meant an abelian group G together with a symmetric bilinear mapping V of $G \times G$ into the additive group of the rationals mod 1 that satisfies the primitivity condition: if $g_1 \not\equiv 0$ there exists $g_2 \not\equiv 0$ such that $V(g_1, g_2) \not\equiv 0 \pmod 1$. A finitely generated V-group is necessarily a finite group. The authors associate to any integral symmetric non-singular matrix $A = (a_{ij})$ a V-group $G(A^{-1})$ defined as follows: $G(A^{-1})$ is generated by elements $g_1, \cdots, g_n$ subject to the defining relations $\sum_j a_{ij} g_j = 0$ $(i = 1, \cdots, n)$; $(V(g_i, g_j)) \equiv A^{-1} \pmod 1$. The main theorem states that two integral symmetric matrices A and B are related by a sequence of the operations $Q_i^{\pm}$ $(i = 1, 2)$, where $Q_1 : A \to S'AS$, S integral and unimodular; $Q_2 : A \to \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}$, if and only if the associated V-groups $G(A^{-1})$ and $G(B^{-1})$ are V-isomorphic. The proof of this depends on a little-known theorem of A. Meyer [Vierteljschr. Naturforsch. Ges. Zürich **36**, 241–250 (1891)] about the number of classes of quadratic forms in a genus and an equally recondite theorem of M. Eichler [Quadratische Formen und orthogonale Gruppen, Springer, Berlin, 1952; these Rev. **14**, 540] about the spinor-representations of the orthogonal group. The complete argument is given only for the case $\det A \not\equiv 0 \pmod 2$. [It seems to the reviewer that this theorem is of considerable importance, not only to knot theory but to the theory of quadratic forms itself; can it be that there is no simple direct proof of it?]

To every regular projection of a knot there is associated [cf. Reidemeister, Knotentheorie, Springer, Berlin, 1932, pp. 25–31] an integral symmetric matrix A. A theorem of Seifert [Abh. Math. Sem. Hamburg. Univ. **11**, 84–101 (1935)] says that $G(A^{-1})$ is then the 1-dimensional homology group H of the second branched cyclic covering of the knot. Since the matrices associated with different projections of the same knot are related by the operations $Q_i^{\pm}$ $(i = 1, 2)$ the relation between the quadratic form xAx' of a knot and the homology group H of the second cyclic covering is completely explained. (Here the authors have simplified the situation in an important way by eliminating the operation $Q_3 : A \to \begin{pmatrix} A & 0 & 0 \\ 0 & a & 1 \\ 0 & 1 & 0 \end{pmatrix}$ [cf. Reidemeister, ibid.] by showing that Q_3 is expressible in terms of Q_1 and Q_2.

R. H. Fox (Princeton, N. J.).

Osima, Masaru. Some remarks on the characters of the symmetric group. Canadian J. Math. **5**, 336–343 (1953).

The author extends some relations between the characters of symmetric groups obtained previously by himself [Math. J. Okayama Univ. **1**, 63–68 (1952); these Rev. **14**, 243] and by J. H. Chung [Canadian J. Math. **3**, 309–327 (1951); these Rev. **13**, 106]. Using results of G. de B. Robinson [Proc. Nat. Acad. Sci. U. S. A. **37**, 694–696 (1951); these Rev. **13**, 530] he gives an explicit formula for the number of modular characters of the symmetric group belonging to a given block. *R. Brauer* (Cambridge, Mass.).

***Aczél, János. Über einige Funktionalgleichungen der Theorie der kontinuierlichen Gruppen.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 565–569. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

The author states and discusses some of his previous results in the theory of functional equations and their connections with continuous groups of transformations [Aczél, Bull. Soc. Math. France **76**, 59–64 (1948); Aczél, Kalmár, and Mikusiński, Studia Math. **12**, 112–116 (1951); these Rev. **10**, 685; **13**, 246]. Some generalisations and new results are also discussed. *P. Erdös* (South Bend, Ind.).

Harish-Chandra. Representations of a semisimple Lie group on a Banach space. I. Trans. Amer. Math. Soc. **75**, 185–243 (1953).

In this paper the author presents details and proofs of some of his fundamental results on infinite-dimensional representations of semisimple Lie groups and Lie algebras [Proc. Nat. Acad. Sci. U. S. A. **37**, 170–173, 362–365, 366–369, 691–694 (1951); these Rev. **13**, 106, 107]. Details of the remaining results announced in these notes are to appear in subsequent papers. This paper is divided into three parts. Part I contains a purely algebraic treatment of infinite-dimensional representations of semisimple Lie algebras which has been given by the author in the special case of complex semisimple Lie algebras previously [Trans. Amer. Math. Soc. **70**, 28–96 (1951); these Rev. **13**, 428; see also Dixmier, Acta Sci. Math. Szeged **14**, 145–156 (1952); these Rev. **14**, 12]. Part II is a study of the relationships between the representations of the Lie algebra and those of the Lie group G. This is much more difficult here than in the known finite-dimensional case and involves a detailed study of what the author calls well behaved vectors ψ. If $x \to \pi(x)$ is the given representation of G in a Banach space H, then ψ is called well behaved if $x \to \pi(x)\psi$ is an analytic mapping of G into H. It is proved that sufficiently many well behaved vectors exist and that every vector can be approximated by well behaved ones in a manner suitable for later applications. Part III combines results of parts I and II and this enables the author to obtain deep results about the representations of semisimple Lie groups. One of

these results asserts that every (weakly measurable and hence strongly continuous) unitary representation of any semisimple Lie group is of type I in the sense of rings of operators, which had been conjectured by the reviewer.

F. I. Mautner (Baltimore, Md.).

Mackey, George W. Induced representations of locally compact groups. II. The Frobenius reciprocity theorem. Ann. of Math. (2) **58**, 193–221 (1953).

In part I [Ann. of Math. (2) **55**, 101–139 (1952); these Rev. **13**, 434] the author began a systematic study of induced representations of locally compact topological groups G and obtained generalizations of the Frobenius reciprocity theorem. In the present paper this study is continued and some of the main results obtained are forms of the generalized Frobenius theorem which deal effectively with continuous direct sums. One such formulation contains as a special case the reviewer's generalization of the Frobenius reciprocity theorem [Amer. J. Math. **74**, 737–758 (1952); these Rev. **14**, 134]. The author's method of approach has as its starting point a new proof of the Frobenius theorem for finite groups which generalizes more significantly to locally compact groups than previous proofs. This generalization involves a detailed study of direct integrals of certain representations and their Kronecker products. The author illustrates his theory by applying it to the 2×2 complex unimodular group and obtains explicit formulas for the decomposition of certain induced representations and Kronecker products. The paper concludes with a brief discussion of possible further developments.

F. I. Mautner (Baltimore, Md.).

Makkeĭ, D. V. [Mackey, G. W.] Functions on locally compact groups. Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 95–129 (1953). (Russian)

Translated from Bull. Amer. Math. Soc. **56**, 385–412 (1950); these Rev. **12**, 588.

Goto, Morikuni. Dense imbedding of topological groups. Proc. Amer. Math. Soc. **4**, 653–655 (1953).

Let G be a locally compact connected group and let $A(G)$ be the group of automorphisms topologized in the usual way. Call G a (CA) group if the group of inner automorphisms $I(G)$ is closed in $A(G)$. The note contains a short proof of the following theorem. Let G be a locally compact connected locally connected (GA) group and H a locally compact group. Assume that the center of Z is compact. If there exists a continuous isomorphism ϕ mapping G into H, then $\phi(G)$ is closed in H. The theorem implies a number of similar results recently obtained by van Est for Lie groups [Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 321–328 (1951); these Rev. **13**, 432].

P. A. Smith (New York, N. Y.).

Mostert, Paul S. Local cross sections in locally compact groups. Proc. Amer. Math. Soc. **4**, 645–649 (1953).

Any closed subgroup H of a topological group G together with its cosets determines a decomposition of the group into homeomorphic sets. The author studies for locally compact groups the question of whether this decomposition is a fibering, that is, whether it is locally a direct product. This is equivalent to the existence of a continuous mapping on an open set of G/H to G which is inverse to the natural mapping of G onto G/H. In general such mappings need not exist, but it is proved that they do exist if G has finite topological dimension. This is, of course, well-known for Lie groups and the proof in general relies on the fact that any connected finite-dimensional topological group is a projective limit of Lie groups all of the same dimension.

A. M. Gleason (Cambridge, Mass.).

Edwards, R. E. On convex spans of translates of functions on a group. Proc. London Math. Soc. (3) **3**, 222–242 (1953).

Let G be a locally compact group. The author deals with a real linear convex topological space E of functions $f, g, \cdots$, defined on G and translation invariant. It is assumed that the mapping $a \to f_a(x) = f(ax)$ is continuous on G to E. The dual space of linear functionals $\phi, \psi, \cdots$, is denoted by E' and the bilinear form by $\langle f, \phi\rangle$. For a given closed set $A \subset G$ and $f \in E$, the closed convex extension of $[f_{a^{-1}}(x); a \in A]$ is denoted by $C(f, A)$. The central problem considered is that of characterizing $C(f, A)$. It is shown that $g \in C(f, A)$ if and only if $\langle g, \phi\rangle \leqq \sup_{a \in A} \langle f_{a^{-1}}, \phi\rangle$ for all $\phi \in E'$. Let $\bar{P}_A$ be the set of Radon measures of total mass $\leqq 1$ and having supports in A and let P_A be the normalized measures in $\bar{P}_A$. It is clear that some of the elements of $C(f, A)$ can be represented by an abstract integral of the form $g = \int f_{a^{-1}} d\mu$ where $\mu \in \bar{P}_A$. In order that the integral exist for arbitrary $\mu \in \bar{P}_A$, conditions on E, f and A must be imposed. It is assumed either (α) that the closed convex extension of any compact subset of E is weakly complete, or (β) that the closed bounded subsets of E are weakly complete. Then if A is compact and if E satisfies (α), each element g of $C(f, A)$ has an integral representation with $\mu \in P_A$. A similar result holds if $C(f, A)$ is bounded and E satisfies (β). We omit a complete listing of the hypotheses; however one can now assert only that $\mu \in \bar{P}_A$ unless θ non-$\in C(f, A)$ in which case $\mu \in P_A$. This motivates a discussion of conditions for the ergodicity of f, that is, for $\theta \in C(f, G)$. In the second half of the paper it is assumed that G is abelian and the above results are translated into relations involving the Fourier transforms of f and g. The paper concludes with an application to positive definite functions having their spectra contained in A.

R. S. Phillips (New Haven, Conn.).

NUMBER THEORY

Franqui, Benito, and Garcia, Mariano. Some new multiply perfect numbers. Amer. Math. Monthly **60**, 459–462 (1953).

de Bruijn, N. G., and Zaring, W. M. On invariants of G.C.D. algorithms. Nieuw Arch. Wiskunde (3) **1**, 105–112 (1953).

Let $a_1 > a_2 > 0$ be integers. Put

$$a_1 = a_2 q_1 + \epsilon_1 a_3, \quad a_2 = a_3 q_2 + \epsilon_2 a_4, \quad \cdots, \quad a_{n-1} = a_n q_{n-1} + \epsilon_{n-1} a_{n+1}, \quad a_n = a_{n+1} q_n,$$

where $a_1 > a_2 > \cdots > a_{n+1} > 0$, $\epsilon_i = \pm 1$. Such a set of equations is called an algorithm for finding the greatest common divisor of a_1 and a_2. Let S be a subset of the set of rationals $x_1 - 1 < x < 1$ such that for each rational number y the congruence $y \equiv x \pmod 1$, $x \in S$, has exactly one solution. We say that our algorithm is an S algorithm if ϵ_i $(i = 1, 2, \cdots, n-1)$ is determined by $\epsilon_i a_{i+2}/a_{i+1} \in S$. If S is the set $0 \leqq x < 1$, we obtain the Euclidean algorithm; if $-\frac{1}{2} < x \leqq \frac{1}{2}$, the least remainder algorithm; if $-1 < x \leqq 0$,

the anti-Euclidean algorithm; and if $-1<x\leqq -\frac{1}{2}$, $\frac{1}{2}\leqq x<1$, the greatest remainder algorithm.

The authors study in detail those algorithms, continuing the previous work of Goodman and Zaring [Amer. Math. Monthly **59**, 156–159 (1952); these Rev. **13**, 912]. Some of their results are: The length of every algorithm is at least as long as the least remainder algorithm (Kronecker) and is shorter than the greatest remainder algorithm. (The number n is defined as the length of an algorithm.) Several other results are proved. *P. Erdös* (South Bend, Ind.).

Ansari, A. R. A theorem on matrices analogous to Fermat's theorem. Proc. Nat. Inst. Sci. India **19**, 465–467 (1953).

Let B be any square matrix of order n with integral elements such that $B^2\equiv I \pmod{2p}$, p being any prime and I the identity matrix. It is proved by induction that, for any positive integer α, $B^{2p^\alpha}\equiv I \pmod{2p^{\alpha+1}}$. *I. Niven.*

Mordell, L. J. On the integer solutions of the equation $x^2+y^2+z^2+2xyz=n$. J. London Math. Soc. **28**, 500–510 (1953).

It is proved that the equation of the title has no solutions in integers if $n\equiv 3 \pmod 4$, $n\equiv 6 \pmod 8$, $n\equiv \pm 3 \pmod 9$, $n=1-4k^2$ with $k\not\equiv 0 \pmod 4$ and k has no prime factors of the form $3+4j$, or $n=1-3k^2$ with $(k,4)=2$, $(k,3)=1$ and k has no prime factors of the form $2+3j$. On the other hand, one solution of the equation implies an infinitude of solutions, except possibly when n is a perfect square having no prime factors of the form $1+4j$. Also there are infinitely many solutions if $n=2^r$ and r is odd, but only the solutions typified by $x=y=0$ when r is even. Proofs are given of A. Hurwitz' theorem that the equation has only the solution $x=y=z=0$ in case $n=0$, and of P. Bachmann's complete parametric solution for the case $n=1$. For any odd n it is established by explicit formulas that the solutions of the equation can be put into one-to-one correspondence with simultaneous solutions of $x^2+y^2+z^2+w^2=m$, $x+y+z+w=2$, with $m=\frac{1}{2}(3n+1)$. *I. Niven* (Eugene, Ore.).

Rédei, L. Die Existenz eines ungeraden quadratischen Nichtrestes mod p im Intervall $1, \sqrt{p}$. Acta Sci. Math. Szeged **15**, 12–19 (1953).

Nagell has proved that if $p\equiv \pm 1 \pmod 8$, $p\neq 7$ or 23, then there exists an odd prime $q<p^{\frac{1}{2}}$ such that $(q/p)=-1$; if $p\equiv 3$ or 5 (mod 8), then there is a $q<2p^{\frac{1}{2}}+1$ or $(2p)^{\frac{1}{2}}$ such that $(q/p)=-1$. In the present paper the following sharper result is obtained. If $p\neq 3, 5, 7, 11, 13, 23, 59, 109, 131$, then there exists an odd prime $q<p^{\frac{1}{2}}$ such that $(q/p)=-1$. (It is noted in proof that part of this theorem had been proved earlier by T. Nagell [Ark. Mat. **1**, 573–578 (1951); these Rev. **14**, 247] and A. Brauer [Math. Z. **33**, 161–176 (1931)].)
L. Carlitz (Durham, N. C.).

Please see the note on p. 1159

Val'fiš, A. Z. Isolated prime numbers. Doklady Akad. Nauk SSSR (N.S.) **90**, 711–713 (1953). (Russian)

Sierpiński introduced the concept of isolated primes, which is to say primes which are sufficiently far from a neighboring prime. Walfisz shows that it is possible for arbitrary k to find sequences of k consecutive primes which are each isolated. *R. Bellman* (Santa Monica, Calif.).

Richert, Hans-Egon. Aus der additiven Primzahltheorie. J. Reine Angew. Math. **191**, 179–198 (1953).

The first part of this paper is concerned with the number of representations, $R(m;s,n)$, of an integer m in the form $m=\sum a_i p_i$, $1\leqq i\leqq s$, $s\geqq 3$, where the a_i are integers, positive or negative but not zero, such that $(a_i,a_k)=1$, $i\neq k$, and the p_i are odd primes $\leqq n$. Let $T(m;s,n)$ denote the number of representations of m in the form $m=\sum a_i x_i$, where the x_i are positive integers $\leqq n$. Following the Vinogradov method, the author proves (Satz 1) that

$$R(m;s,n)=(S(m)/\log^s n)T(m;s,n)+O(n^{s-1}/\log^{s+1}n),$$

where $S(m)$, a purely arithmetic function, is the so-called "singular series". A second result (Satz 2) generalizes the Vinogradov result for three primes. It is shown that, if all the a_i are positive, then

$$R(n;s,n)=\frac{S(n)}{(s-1)!a_1\cdots a_s}\cdot\frac{n^{s-1}}{\log^s n}+O\left(\frac{n^{s-1}}{\log^{s+1}n}\right)$$

The author has apparently overlooked a paper by the reviewer and H. Weyl [Amer. J. Math. **64**, 539–552 (1942); these Rev. **4**, 35] in which the first result is implied and the second explicitly stated.

The second part deals with $G(m;n)$, the number of representations of m in the form $m=p_1+p_2$, $3\leqq p_i\leqq n$. The original Goldbach conjecture was $G(m;m)>0$ for all even $m\geqq 6$. Since the work of Hardy and Littlewood it has been conjectured that, for even m, $G(m;m)\sim S(m)m/\log^2 m$, and the author shows, again by the Vinogradov method, that, on the average, this is true. More precisely, he proves (Satz 5) that $(1/2u)\sum G(m;n)$, $n-u\leqq m\leqq n+u$, $u=n/\log^\beta n$, $\beta>0$, and $(1/u)\sum G(m;m)$, $n-u\leqq m\sim n$, are each equal to $n/\log^2 n+O(n\log\log n/\log^3 n)$. A similar result (Satz 6) holds for

$$(k/n)\sum G(m;m),\quad m\leqq n,\quad m\equiv h \pmod k,$$
$$1\leqq h\leqq k\leqq \log^\alpha n,\quad \alpha\geqq 25,$$

but with an arithmetic function depending on the primes which divide k and h as a coefficient of $n/\log^2 n$ and an error term which is not quite as good. *R. D. James.*

Iseki, Kanesiroo. A remark on the Goldbach-Vinogradov theorem. Proc. Japan Acad. **25**, 185–187 (1949).

This paper sketches a proof of the asymptotic formula for the number of solutions in primes p_1, p_2, p_3 of the equation $n=p_1+p_2+p_3$ where $p_m\equiv r_m \pmod k$ for $m=1, 2, 3$ and the integers r_m are fixed and such that $(r_m,k)=1$; also, it is required that $n\equiv r_1+r_2+r_3 \pmod k$. The result is the same as that obtained in the more detailed later paper of Ayoub [Trans. Amer. Math. Soc. **74**, 482–491 (1953); these Rev. **14**, 847]. *L. Schoenfeld* (Urbana, Ill.).

Cugiani, Marco. Sugli intervalli fra i valori dell'argomento pei quali un polinomio risulta libero da potenze. Rivista Mat. Univ. Parma **4**, 95–103 (1953).

If an integer does not have a kth power other than 1 for a divisor, it is said to be k-free ($k\geqq 2$). Let $F(x)$ be an integral-valued polynomial of degree g with rational coefficients and discriminant different from zero. For $g\geqq 1$ and a fixed integer $l\geqq 1$ put $k=g+l$. The author considers sequences of integers $1\leqq q_1<q_2<\cdots<q_n<\cdots$ which are such that the corresponding polynomial values $F(q_1),\cdots,F(q_n),\cdots$ are all k-free integers. Halberstam and Roth [J. London Math. Soc. **26**, 268–273 (1951); these Rev. **13**, 208] proved in the particular case $F(x)=x$ (q_n is now the nth k-free integer) that $\delta_n=O(n^{(1/2k)+\epsilon})$, where $\delta_n=q_{n+1}-q_n$. Employing only elementary arguments in conjunction with the method of Viggo Brun the author obtains preliminary results about the asymptotic behaviour of δ_n in the general case. Let $\nu(m)$ denote the number of solutions of the congruence $F(x)\equiv 0$

(mod m). For a fixed integer $s \geqq 2$ put $\Delta_s = \prod(1-\nu(p^s)/p^s)$, where the product extends over all the primes. Then the author establishes the following three results: (a) $\liminf \delta_n \leqq 1/\Delta_s$; (b) $\limsup \delta_n \log\log n/\log n \geqq h/gk$; (c) $\delta_n = O(n^{(g/k)-\alpha}/(\log n)^{\beta})$, where $\alpha = g/(k^2+k)$, $\beta = k/(k+1)$ for $g \leqq l$, and $\alpha = l/(k^2-k)$, $\beta = 1$ for $g > l$.

A. L. Whiteman (Princeton, N. J.).

Sathe, L. G. On a problem of Hardy on the distribution of integers having a given number of prime factors. I. J. Indian Math. Soc. (N.S.) **17**, 63–82 (1953).

This is the first part of a paper in which the author proves (among others) the following results: Denote by $\pi_\nu(x)$ the number of squarefree integers having ν prime factors. Put $\nu = k \log\log x$. Let $k < e$. Then

$$\pi_\nu(x) = (1+o(1))f(k)\frac{x}{\log x}\frac{(\log\log x)^{\nu-1}}{(\nu-1)!}$$

where (p runs through all primes)

$$f(k) = \frac{1}{\Gamma(k+1)}\left[\prod_p\left(1-\frac{1}{p}\right)e^{1/p}\right]^k\prod_p\left(1+\frac{k}{p}e^{-k/p}\right).$$

Similar asymptotic formulas are obtained for $\sigma_\nu(x)$ and $\rho_\nu(x)$, where $\sigma_\nu(x)$ denotes the number of integers $\leqq x$ having ν prime factors, multiple factors counted multiply, and in $\rho_\nu(x)$ multiple factors are counted only once.

The proofs are very complicated. These results strengthen and supersede all previous ones and were found about simultaneously with those of the reviewer [Ann. of Math. (2) **49**, 53–66 (1948); these Rev. **9**, 333] which are special cases ($\nu = \log\log x + c(\log\log x)^{1/2}$) of those of the author.

P. Erdös (South Bend, Ind.).

***Kubilyus, I. P., and Linnik, Yu. V On the decomposition of the product of three numbers into the sum of two squares.** Trudy Mat. Inst. Steklov., v. 38, pp. 170–172. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

The authors prove that there are infinitely many positive integers w such that (1) w is the product of three prime numbers of the form $4n+1$, (2) $w = k^2+l^2$, where k and l are integers and $|l| \leqq \exp(\ln w)^{4/5}$. This is a corollary of a more inclusive theorem derived in this paper from certain results in Vinogradov's monograph [Trav. Inst. Math. Stekloff **23** (1947); these Rev. **10**, 599].

P. T. Bateman.

Mardžanišvili, K. K. On the simultaneous representation of pairs of numbers by sums of prime numbers and their squares. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 183–208 (1951). (Russian. Georgian summary)

Suppose ϵ is a given positive number $< \frac{1}{2}$ and s is a given positive integer $\geqq 7$. The author proves in full that the system of Diophantine equations (*) $p_1+p_2+\cdots+p_s = N_1$, $p_1^2+p_2^2+\cdots+p_s^2 = N_2$ is solvable in prime numbers $p_1, p_2, \cdots, p_s$ provided N_1 and N_2 are sufficiently large positive integers satisfying the requirements $1+\epsilon \leqq N_1N_2^{-1} \leqq s^1-\epsilon$, $N_2 \equiv N_1 \pmod 2$, $N_2 \equiv s \pmod{24}$. This is done in the usual two steps. First, he proves that if N_1 and N_2 are any positive integers such that $1+\epsilon \leqq N_1N_2^{-1} \leqq s^1-\epsilon$, then the number of solutions of (*) in prime numbers is given by

$$B(N_1N_2^{-1}, s)N_2^{(s-2)/2}(\log N_2)^{-s}\mathfrak{S}_s(N_1, N_2) + O(N_2^{(s-2)/2}(\log N_2)^{-s-\omega}),$$

where ω is a certain positive constant,

$$0 < C_1(s, \epsilon) < B(N_1N_2^{-1}, s) < C_2(s, \epsilon),$$

and $\mathfrak{S}_s(N_1, N_2)$ is the singular series. [This first part of the argument, which is based on Vinogradov's method, can also be found in Hua's monograph, The additive theory of prime numbers, Trav. Inst. Math. Stekloff **22** (1947), Chap. 10; these Rev. **10**, 597.] Second, he shows that if $N_2 \equiv N_1 \pmod 2$ and $N_2 \equiv s \pmod{24}$, then $\mathfrak{S}_s(N_1, N_2) > C(s) > 0$. The resumé of the author's doctoral dissertation [Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 236–240 (1950); these Rev. **11**, 502] seems to indicate that the present paper is essentially the last chapter of that apparently unpublished work. Also the author has asserted in an expository article [Acta Math. Acad. Sci. Hungar. **2**, 223–227 (1951); these Rev. **15**, 13] that he first obtained the above result in 1942.

P. T. Bateman (Urbana, Ill.).

Lomadze, G. A. On the representation of numbers by sums of squares. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **16**, 231–275 (1948). (Russian. Georgian summary)

This is a readable, semi-expository paper in which the author works out in detail the explicit formulas for $r_s(n)$, the number of representations of a positive integer n as a sum of s squares, when $9 \leqq s \leqq 24$. The method is based on the rudiments of the theory of modular functions and stems from the classical papers of Mordell [Quart. J. Pure Appl. Math. **48**, 93–104 (1917); Trans. Cambridge Philos. Soc. **22**, 361–372 (1919)] and Hardy [Trans. Amer. Math. Soc. **21**, 255–284 (1920)]. [See also Dickson, Studies in the theory of numbers, Chicago, 1930, Chap. 13, and Hardy, Ramanujan . . . , Cambridge, 1940, Chap. 9; these Rev. **3**, 71.] Actually the author follows a variant of the Hardy-Mordell method given by Estermann [Acta Arith. **2**, 47–79 (1936)] which avoids the use of the theory of modular functions as such. The basic result is that if $s \geqq 5$, then in the upper half of the τ-plane

$$\sum_{n=1}^{\infty} r_s(n)e^{\pi i \tau n} = \sum_{n=1}^{\infty}\rho_s(n)e^{\pi i \tau n} + \sum_{\nu=1}^{l}\alpha_s^{(\nu)}\vartheta_3(0|\tau)^{s-3\nu}\vartheta_0(0|\tau)^{4\nu}\vartheta_2(0|\tau)^{4\nu}$$

where $\rho_s(n)$ is Hardy's approximation to $r_s(n)$, $l = [(s-1)/8]$, and the $\alpha_s^{(\nu)}$ are certain constants. From this the detailed results for each s from 9 to 24 are derived by evaluating the $\alpha_s^{(\nu)}$ numerically and using Hardy's formulas for $\rho_s(n)$.

P. T. Bateman (Urbana, Ill.).

Parthasarathy, M. On the representation of an integer as the sum of three fourth powers. Proc. Amer. Math. Soc. **4**, 523–527 (1953).

Let $r_{3,4}(N)$ denote the number of representations of N as the sum of three fourth powers of positive integers. It is proved by S. Chowla [J. Indian Math. Soc. **20**, 121–128 (1934)] that, for infinitely many N,

$$r_{3,4}(N) > c_1 \log N/\log\log N.$$

This estimate is improved in the paper under review to

$$r_{3,4}(N) > \exp(c_2 \log N/\log\log N)$$

for infinitely many N. The argument is elementary and depends on estimates of the number of representations of N in the form x^2+3y^2 ($x > 7y > 0$) and also in the form x^2-xy+y^2 ($x > 4y > 0$). The reviewer observes that the term $\xi^2-\eta^2$ in the equation (3.1) should be $(\xi^2-\eta^2)^4$.

L. Mirsky (Sheffield).

Moser, L., and Lambek, J. On monotone multiplicative functions. Proc. Amer. Math. Soc. **4**, 544–545 (1953).

Let $f(n)$ be a real-valued function defined for the positive integers. Assume that $f(m\cdot n)=f(m)\cdot f(n)$ for $(m,n)=1$, i.e., $f(n)$ is weakly multiplicative. The authors prove that if $f(m)\geqq f(n)$ for $m\geqq n$ and $f(n)\not\equiv 0$, then $f(n)=n^k$ where k is a constant. The authors overlooked that this was proved by the reviewer [Ann. of Math. (2) **47**, 1–20 (1946), p. 17; these Rev. **7**, 416]. The proof of the authors is completely different and very much simpler than my own.

P. Erdös (South Bend, Ind.).

Freĭman, G. A. On the exponential density of sequences. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 385–388 (1952). (Russian)

The exponential density of an increasing sequence S of positive integers is $\liminf_{x\to\infty}(\log\pi(x))/(\log x)$, where $\pi(x)$ is the number of terms of S which do not exceed x. Let A be a given increasing sequence $a_1, a_2, \cdots$ of positive integers and let η_k be the exponential density of kA, the increasing sequence formed by those positive integers which are expressible in the form $a_{i_1}+a_{i_2}+\cdots+a_{i_k}$. The author proves several inequalities for η_k under various restrictions on A and shows that these inequalities are best possible. For example, he proves that if $a_i-a_{i-1}=O(a_i^{\sigma})$, then $\eta_k\geqq 1-\sigma^k$. All the proofs are elementary. *P. T. Bateman.*

Kneser, Martin. Abschätzung der asymptotischen Dichte von Summenmengen. Math. Z. **58**, 459–484 (1953).

It is well known that the analogon to the Fundamental Theorem on the density of sums of sets of positive integers does not hold for the asymptotic density. A counter exampls is given by the sets A, B where A consists of all numbere congruent to $1, \cdots, l-1 \pmod g$, B of all numbers congruent to $1, \cdots, k-1 \pmod g$ where $k+l\leqq g$. Further counterexamples can be obtained by deleting from A and B sets which are sufficiently small. The author's main result means that these are essentially all cases which do not satisfy an analogon to the density theorem. We shall write $A\sim B$ if A and B coincide for sufficiently large numbers. The author proves: If $A_0, \cdots, A_n$ are $n+1$ sets and $\sum A_i=A$, then either $\liminf_{x\to\infty} A(x)/x>\liminf_{x\to\infty}\sum_0^n A_i(x)/x$ or there exists a $g>0$ and congruence sets mod g: $A_0^{(g)}, \cdots, A_n^{(g)}$, such that $A\sim A^{(g)}$, $A_i^{(g)}\supset A_i$, and

$$\lim_{x\to\infty}\frac{C^{(g)}(x)}{x}=\lim_{x\to\infty}\frac{\sum_0^n A_i(x)}{x}-\frac{n}{g}.$$

The application of this result and of the methods used in obtaining it yields new estimates for the asymptotic density of sums of sets containing the well known result of Erdös [Ann. of Math. (2) **43**, 65–68 (1942); these Rev. **3**, 165] and an improvement of Erdös' result due to Ostmann [J. Reine Angew. Math. **187**, 183–188 (1950); these Rev. **11**, 646]. *H. B. Mann* (Columbus, Ohio).

Lorentz, G. G. Multiplicity of representation of integers by sums of elements of two given sets. J. London Math. Soc. **28**, 464–467 (1953).

The author proves among others the following theorems. 1) Let A and B be two infinite sequences of integers, assume that the lower density $d(A)$ of A is $\alpha>0$. Then there exists an infinite sequence n_i of asymptotic density $\geqq\alpha$ for which $\rho(n_i)\to\infty$ ($\rho(n)$ denotes the number of solutions of $n=a+b$, $a\in A$, $b\in B$). 2) Let $d(A)=\alpha$, $d(B)=\beta$. Then

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{\rho(k)}{k}\right)\geqq\alpha\beta.$$

P. Erdös (South Bend, Ind.).

Lech, Christer. A note on recurring series. Ark. Mat. **2**, 417–421 (1953).

The author proves the following theorem: In a field of characteristic 0 let a sequence c_ν, $\nu=0, 1, 2, \cdots$, satisfy the recursion formula

$$c_\nu=a_1c_{\nu-1}+a_2c_{\nu-2}+\cdots+a_nc_{\nu-n},\quad \nu=n, n+1, n+2, \cdots.$$

Then if $c_\nu=0$ for infinitely many ν, those ν for which $c_\nu=0$ form a periodic sequence for $\nu>\nu_0$. The author shows by an example that the result does not hold for fields of characteristic p. He also points out that for the field of rational numbers this result is due to Skolem and for the field of algebraic numbers to Mahler. *P. Erdös.*

Erdös, P. Arithmetical properties of polynomials. J. London Math. Soc. **28**, 416–425 (1953).

Let $f(x)$ denote a polynomial of degree l whose coefficients are integers with highest common factor 1 and with positive highest coefficient. The writer first notes the known result that there are infinitely many positive integers n such that $f(n)$ is lth power free. He then proves that if $l\geqq 3$ then there are infinitely many positive integers n such that $f(n)$ is $(l-1)$th power free. It is necessary to exclude an exceptional case; this is done by assuming that when l is a power of 2 there exists an n such that $f(n)\not\equiv 0 \pmod{2^{l-1}}$. In the exceptional case it follows from the proof of the main result that there are infinitely many n for which $f(n)=2^{l-1}u_n$, where u_n is odd and $(l-1)$th power free. It is remarked in conclusion that the method of proof of the present paper should allow one to prove that every sufficiently large positive integer is the sum of an lth power and an $(l-1)$th power-free integer. *L. Carlitz* (Durham, N. C.).

Carlitz, L. Note on the class number of real quadratic fields. Proc. Amer. Math. Soc. **4**, 535–537 (1953).

Let p be a prime $\equiv 1 \pmod 4$ and let $h(p)$ denote the class number of the real quadratic field $R(p^{1/2})$; let $\epsilon=\frac{1}{2}(t+up^{1/2})$ denote the fundamental unit of the field ($\epsilon>1$). Ankeny, Artin and Chowla [Proc. Nat. Acad. Sci. U. S. A. **37**, 524–525 (1951); Ann. of Math. (2) **56**, 479–493 (1952); these Rev. **13**, 212; **14**, 251] proved the following results: (1) $4uh/t\equiv -\sum_{r=1}^{(p-1)/4}(r/p)/r \pmod p$ ($p\equiv 5 \pmod 8$), where (r/p) is the Legendre symbol; (2) $uh/t\equiv B_{(p-1)/2} \pmod p$, where B_m denotes a Bernoulli number in the even suffix notation; (3) $2uh/t\equiv(A+B)/p \pmod p$, where A is the product of the quadratic residues of p between 0 and p, and B is the product of the nonresidues of p between 0 and p. In this note the author points out that if we assume (2), then (1) and (3) can be proved quite simply. The proofs make use of well known relations involving Bernoulli numbers and polynomials [cf. N. Nielsen, Traité élémentaire des nombres de Bernoulli, Gauthier-Villars, Paris, 1923, Chapters 14 and 20; N. E. Nörlund, Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924, p. 22]. *A. L. Whiteman* (Princeton, N. J.).

Carlitz, L. Some sums containing Bernoulli functions. Amer. Math. Monthly **60**, 475–476 (1953).

Nicol, Charles A. On restricted partitions and a generalization of the Euler ϕ number and the Moebius function. Proc. Nat. Acad. Sci. U. S. A. **39**, 963–968 (1953).

The author considers the coefficients of the powers of x appearing in the finite product $F_{n-1}(z,x)=\prod_{r=1}^{n-1}(z-x^r)$, and states without proof a number of results concerning the arithmetic properties of these coefficients. Of particular interest are the connections between these coefficients and the Ramanujan sums. Proofs will be published separately.

R. Bellman (Santa Monica, Calif.).

Schoeneberg, Bruno. Über den Zusammenhang der Eisensteinschen Reihen und Thetareihen mit der Diskriminante der elliptischen Funktionen. Math. Ann. **126**, 177–184 (1953).

Making use of the work of Hecke on Eisenstein series and theta series, the writer shows how to construct the principal function of the congruence group of genus $p=0$, when the number of corners of the fundamental domain $\geqq 3$, by means of Eisenstein series. He also exhibits the functions $\Delta_\lambda(\tau)=(\Delta(\lambda))^{1/\lambda}$, $\lambda\,|\,12$, in terms of Eisenstein series. Define the generalized binary theta series

$$\vartheta_k(\tau;\rho,\mathfrak{a},QD^{\frac{1}{2}})=\sum\mu^{k-1}\exp\{2\pi i\tau\mu\mu'/AQ|D|\},$$

the summation extending over all integers of $R(D^{\frac{1}{2}})$ such that $\mu\equiv\rho \pmod{\mathfrak{a}QD^{\frac{1}{2}}}$, where $\rho\in\mathfrak{a}$ and A is the norm of $\mathfrak{a}$. Then $\Delta_\lambda(\tau)$ is represented by means of ϑ_k in such a manner as to obtain the Euler product of the associated Dirichlet series and also to get bounds for the coefficients. Finally the writer indicates a connection between the number of representations of an integer as a sum of $2k$ squares and the Eisenstein series and the functions $\Delta_\lambda(\tau)$. *L. Carlitz.*

***Čudakov, N. G., and Pavlyučuk, A. K. On summation functions of characters of numerical groups with a finite basis.** Trudy Mat. Inst. Steklov., v. 38, pp. 366–381. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

Suppose $\mathfrak{G}$ is a multiplicative group of positive algebraic numbers with a finite basis $\omega_1,\omega_2,\cdots,\omega_p$, where we assume $\omega_k>1$ for $k=1,2,\cdots,p$. Let $\mathfrak{S}$ be the semigroup generated by $\omega_1,\omega_2,\cdots,\omega_p$. Suppose χ is a (not necessarily bounded) character of $\mathfrak{G}$, that is a homomorphism of $\mathfrak{G}$ into the multiplicative group of non-zero complex numbers. Let H be the function defined on the non-negative real numbers by the formula $H(x)=\sum_{\alpha\in\mathfrak{S},\,\alpha\leqq\exp x}\chi(\alpha)$. Suppose $\sigma_0=\max_{1\leqq k\leqq p}(\log|\chi(\omega_k)|)/(\log\omega_k)$ and q is the number of values of k for which $\sigma_0=(\log|\chi(\omega_k)|)/(\log\omega_k)$. Then the authors prove the following assertions about the behavior of $H(x)$ as $x\to\infty$. (1) If $\sigma_0<0$, then $H(x)$ is bounded. (2) If $\sigma_0=0$, $q=1$, and $\chi(\omega_k)\neq 1$ for all k, then $H(x)$ is bounded. (3) If $\sigma_0=0$ and $\chi(\omega_k)=1$ for some k, then $H(x)=\Omega(x)$. (4) If $\sigma_0=0$ and $q\geqq 2$, then $H(x)=\Omega((\log\log\log x)^{\frac{1}{2}})$. (5) If $\sigma_0>0$, then $H(x)=\Omega(x^{-1}e^{\sigma_0 x})$. Although (1) and (2) are almost trivial, the proofs of (3), (4), and (5) require delicate methods from analytic number theory due to Vinogradov, Gelfond, and Linnik. A particular case of the above is a result of Čudakov and Linnik [Doklady Akad. Nauk SSSR (N.S.) **74**, 193–196 (1950); these Rev. **12**, 393] to the effect that if $\mathfrak{G}$ is a multiplicative subgroup of the positive rationals generated by a finite set $\mathfrak{P}$ of prime numbers and if χ is a bounded character of $\mathfrak{G}$, then $H(x)$ is bounded if and only if $\mathfrak{P}$ contains exactly one prime number and χ is not the principal character of $\mathfrak{G}$. *P. T. Bateman.*

Bredihin, B. M. On characters of numerical semigroups with a sufficiently sparse base. Doklady Akad. Nauk SSSR (N.S.) **90**, 707–710 (1953). (Russian)

Suppose $\mathfrak{G}$ is a multiplicative group of positive real numbers with a denumerable basis $\omega_1,\omega_2,\cdots$, where we assume that $\omega_k>1$ for all k and that for any positive number x there is only a finite number $\pi(x)$ of positive integers k such that $\omega_k\leqq\exp x$. Suppose χ is a bounded character of $\mathfrak{G}$, that is, a homomorphism of $\mathfrak{G}$ into the multiplicative group of complex numbers of absolute value 1. Let $\mathfrak{S}$ be the semigroup generated by $\omega_1,\omega_2,\cdots$, put $H(x)=\sum_{\alpha\in\mathfrak{S},\,\alpha\leqq\exp x}\chi(\alpha)$, and let $\exp_m$ and $\log_m$ denote the mth iterates of the exponential and logarithmic functions respectively. Then the author proves the following assertions about the behavior of $H(x)$ as $x\to\infty$. (1) If $\pi(x)<\eta\log x$ for large x, where η is a positive constant $<(\log 2)^{-1}$, and if $\chi(\omega_k)=1$ for some k, then $H(x)=\Omega(x^{1-\eta\log 2})$. (2) If $\pi(x)<\delta\log_{m+1}x$ for large x, where δ is a positive constant $<(\log 4)^{-1}$, if $\chi(\omega_k)\neq 1$ for all k, and if there exists a pair of basis elements ω_k and ω_l such that the continued fraction of $(\log\omega_k)/(\log\omega_l)$ has the properties $a_\nu=O(\exp_m c\nu)$ and $q_{\nu+1}=O(\exp q_\nu^\rho)$, where a_ν is the νth partial quotient, q_ν is the denominator of the νth convergent, c is a positive constant, and ρ is a positive constant <1, then $H(x)=\Omega(\log_m x)^{1-\delta\log 2}$. The proof of (1) is easy, but the proof of (2) is like the proofs of the latter parts of the theorem of the paper reviewed above. *P. T. Bateman.*

***Suetuna, Zyoiti. Kaisekiteki seisûron. [Analytical theory of numbers.]** Iwanami Shoten, Tokyo, 1950. iii+ii +295+ii pp. 450 yen.

The book under review was originally published some fifteen years ago in the form of lecture notes; the present revised one incorporates recent developments of the theory. The manuscript was, however, in the printer's hand when various elementary proofs were published of the Prime Number theorem and Dirichlet's theorem, thus necessitating their omissions. This book, based mainly on the Riemann ζ-functions and L-functions, is a unique exposition of the analytical theory of numbers in a modern sense as can be seen from the chapter headings: I) Riemann's ζ-functions; II) Hecke's L-functions; III) Dirichlet's L-functions; and IV) Artin's L-series. In order to indicate the importance of L-functions the book concludes with an appendix entitled "A general divisor problem," which was treated by H. Hasse and Z. Suetuna [J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **2**, 133–154 (1931)]. *S. Ikehara* (Tokyo).

Barnes, E. S. Note on non-homogeneous linear forms. Proc. Cambridge Philos. Soc. **49**, 360–362 (1953).

Let θ be an algebraic number of degree n, with conjugates $\theta^{(1)},\cdots,\theta^{(n)}$ of which the first r are real and the last $2s$ are arranged in complex conjugate pairs $\theta^{(r+j)}$, $\theta^{(r+s+j)}$ $(j=1,\cdots,s)$. Define the linear forms

$$L_i(u_1,\cdots,u_n)=u_1\omega_1^{(i)}+\cdots+u_n\omega_n^{(i)}\quad(i=1,\cdots,n),$$

where $\omega_1,\cdots,\omega_n$ is a basis for the integers of $k(\theta)$. Let $c_1,\cdots,c_r$ be real, c_{r+j} and c_{r+s+j} complex conjugates $(j=1,\cdots,s)$, and call M_c the lower bound of $|(L_1-c_1)\cdots(L_n-c_n)|$ for integral $u_1,\cdots,u_n$. Davenport [Quart. J. Math., Oxford Ser. (2) **3**, 32–41 (1952); these Rev. **13**, 918] proved for any set of numbers $c_1,\cdots,c_n$ that $M_c<C_n|d|^{n/2(n-s)}$, where C_n depends only on n, and d is the discriminant of $k(\theta)$. In the present paper, the author uses a method of Tschebotareff to give a very simple proof of Davenport's result.

I. Reiner (Urbana, Ill.).

Davenport, H. On the product of n linear forms. Proc. Cambridge Philos. Soc. **49**, 190–193 (1953).

The author uses Barnes' methods [see the preceding review] to improve the above-stated result to

$$M_s < C_n |d|^{(2n-2)/(2n-2s-1)}$$

for $s>0$, which gives a better estimate except when $r=0$.

I. Reiner (Urbana, Ill.).

Sawyer, D. B. The minima of indefinite binary quadratic forms. J. London Math. Soc. **28**, 387–394 (1953).

A geometric proof is given of a result of Segre [Duke Math. J. **12**, 337–365 (1945); these Rev. **6**, 258]. For the indefinite form $a\xi^2+b\xi\eta+c\eta^2$ of discriminant D, integers ξ, η not both zero exist such that $-\lambda\phi_0(\lambda) \leqq f/D^{1/2} \leqq \lambda^{-1}\phi_0(\lambda)$, where $\lambda>0$ and $\phi_0(\lambda)=\min[(4+\lambda^2)^{-1/2}, (4+\lambda^{-2})^{-1/2}]$. Strict inequalities hold unless λ^2 is integral and f is equivalent to a multiple of $f_0=\xi^2+\lambda^2\xi\eta-\lambda^2\eta^2$. The author also proves that if f is not equivalent to kf_0, then ϕ_0 may be replaced by $(4+(\lambda+\lambda^{-2})^2)^{-1/2}$ and strict inequality holds unless $\lambda=1$ and f is equivalent to a multiple of $\xi^2+2\xi\eta-\eta^2$.

L. Tornheim (Ann Arbor, Mich.).

Mordell, L. J. Note on Sawyer's paper "The product of two non-homogeneous linear forms". J. London Math. Soc. **28**, 510–512 (1953).

Sawyer [same J. **23**, 250–251 (1948); these Rev. **10**, 355] made the unproved assertion that for a given non-homogeneous lattice Λ admissible for the region $|xy| \leqq k$, there exists a quadrilateral with vertices in Λ each in a different quadrant and containing no other points of Λ. A simple geometrical proof is given that either this is the case or there is a triangle of the same type and containing the origin. Sawyer's proof is modified to take care of the second case. Reference is made to Delone [Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 505–538 (1947); these Rev. **9**, 334] who proved there is a parallelogram satisfying the assertion of Sawyer.

L. Tornheim (Ann Arbor, Mich.).

Jones, Burton W., and Hadlock, E. H Properly primitive ternary indefinite quadratic genera of more than one class. Proc. Amer. Math. Soc. **4**, 539–543 (1953).

L'auteur démontre que les genres d'un certain type de formes quadratiques ternaires indéfinies contiennent plus d'une classe. Il en donne, également, des exemples. Les démonstrations sont trop techniques pour pouvoir être résumées ici.

M. Krasner (Paris).

Oppenheim, A. Value of quadratic forms. III. Monatsh. Math. **57**, 97–101 (1953).

The following theorem has been proven by the author in a previous paper [Quart. J. Math., Oxford Ser. (2) **4**, 60–66 (1953); these Rev. **14**, 955]: If f represents zero for integers x_i not all zero, if f is not a multiple of a rational form and if $n \geqq 5$, then the inequalities $0<f(x_1, x_2, \cdots, x_n)<\epsilon$ are solvable for integers $x_1, \cdots, x_n$ for every positive ϵ. For $n=2$ the theorem is false; for $n \geqq 3$ its proof can be reduced to that of the same statement for the particular quadratic form $Q=x_1x_2+\vartheta x_3^2+c_4x_4^2+\cdots+c_nx_n^2$ with ϑ irrational and $c_4, \cdots, c_n$ non-zero integers. In the present paper the theorem is proven for $n=4$; the case $n=3$ is left unsettled. In the proof use is made of (i) the equidistribution (mod 1) of ϑp_n^2 for primes p_n; and (ii) the fact that an integral binary quadratic form represents the numbers $\sigma^2 p$, where σ is a fixed integer, while p runs through the primes of an arithmetic progression. (i) follows from Weyl's theorem on equidistribution (mod 1) [Math. Ann. **77**, 313–352 (1916)] and a paper by Vinogradov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **17**, 165–166 (1937)], while (ii) is proven using the properties of integral binary quadratic forms.

E. Grosswald (Philadelphia, Pa.).

Rogers, K. The minima of some inhomogeneous functions of two variables. J. London Math. Soc. **28**, 394–402 (1953).

Let l, m be two lines through the origin dividing the plane into four quadrants. Let K be a closed point set, symmetric in 0, containing $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$, and such that the part of each quadrant not in K is convex. Then for every (x_0, y_0) there is in K a $(x, y) \equiv (x_0, y_0)$ (mod 1). As a consequence, for a certain class of functions $f(x, y)$ described by geometric properties and for any (x_0, y_0) there exists $(x, y) \equiv (x_0, y_0)$ (mod 1) such that

$$|f(x, y)| \leqq \max[f(0, \tfrac{1}{2}), f(\tfrac{1}{2}, 0), \min\{f(\tfrac{1}{2}, \tfrac{1}{2}), f(\tfrac{1}{2}, -\tfrac{1}{2})\}].$$

An example of such an f is $|\xi\eta|$ where $\xi=ax+by$, $\eta=cx+dy$ $(ad-bc\neq 0)$ and for it the result had been proved by Barnes [Quart. J. Math., Oxford Ser. (2) **1**, 199–210 (1950); these Rev. **13**, 16]. Other examples given are $|\xi\eta|(|\xi|+|\eta|)$ and $|\xi^n\eta^n|$.

L. Tornheim (Ann Arbor, Mich.).

Šapiro-Pyateckiĭ, I. I. On a generalization of the notion of uniform distribution of fractional parts. Mat. Sbornik N.S. **30(72)**, 669–676 (1952). (Russian)

Suppose $\theta_1, \theta_2, \cdots$ is a sequence of real numbers in the interval $[0, 1)$. If $0 \leqq \alpha < \beta \leqq 1$, denote by $P(N; \alpha, \beta)$ the number of positive integers k such that $\alpha \leqq \theta_k \leqq \beta$ and $k \leqq N$. The author calls the sequence $\theta_1, \theta_2, \cdots$ almost uniformly distributed if there exists an increasing sequence of positive integers $N_1, N_2, \cdots$ such that $\lim_{m\to\infty} N_m^{-1}P(N_m; \alpha, \beta) = \beta-\alpha$ for every pair α, β. (Uniform distribution itself refers to the case in which $N_m=m$ for every m.) The author obtains two separate sufficient conditions on a sequence of positive integers $n_1, n_2, \cdots$ under either of which the sequence of fractional parts $\{\alpha n_1\}, \{\alpha n_2\}, \cdots$ is almost uniformly distributed for all but denumerably many real α. Condition I: There exist two sequences of positive integers $m_1, m_2, \cdots$ and $\nu_1, \nu_2, \cdots$ such that $n_k=m_{\nu_k}$ for every k, the sequence $\nu_1, \nu_2, \cdots$ has positive density, and the sequence of fractional parts $\{\alpha m_1\}, \{\alpha m_2\}, \cdots$ is uniformly distributed for all but denumerably many α. Condition II: There exists a positive integer r such that $\int_0^1 |\sum_{k=1}^s \exp(2\pi i x n_k)|^{2r} dx = O(s^{2r}/n_s)$. Each of these two conditions contains the condition of positive density as a special case, the former when $m_k=k$ for every k and the latter when $r=1$. However, the author shows by an example that even the condition of positive density of the sequence $n_1, n_2, \cdots$ is not enough to ensure uniform distribution of the sequence $\{\alpha n_1\}, \{\alpha n_2\}, \cdots$ for all but denumerably many α. The proofs in this paper are not difficult, but they require fairly powerful tools from the theory of functions of a real variable. Needless to say, the basic idea on which Weyl's treatment of uniform distribution is based [Math. Ann. **77**, 313–352 (1916)] also plays a rôle.

P. T. Bateman (Urbana, Ill.).

Kasch, Friedrich. Zur Annäherung algebraischer Zahlen durch arithmetisch charakterisierte rationale Zahlen. Math. Nachr. **10**, 85–98 (1953).

In the well-known theorem of Th. Schneider [J. Reine Angew. Math. **188**, 115–128 (1950); these Rev. **12**, 483] concerning the rate of growth of the denominators in a sequence of rational numbers p_i/q_i which provide sufficiently

good approximations to a given algebraic number, it is assumed that these denominators are essentially products of powers of a fixed set of primes. In the present work, an analogous result is obtained for the case that the q's are essentially factorials. Suppose that $(p_i, q_i)=1$, $q_i \leqq q_{i+1}$, and that $q_i=q_i'q_i''$, where $q_i''f_i=l_i!$ for some integers f_i and l_i, and

$$\limsup_{i\to\infty}\frac{\log q_i'}{\log q_i}=\omega_1, \quad \limsup_{i\to\infty}\frac{\log f_i}{\log q_i}=\omega_2.$$

Then if for some algebraic number α, the inequality $|\alpha-p_i/q_i|<q_i^{-\mu}$ holds for all i, where

$$\mu>1+\omega_1+\omega_2((1-\omega_1)(2+\omega_2)+\omega_1^2),$$

then $\limsup_{i\to\infty}(\log q_{i+1})/(\log q_i)=\infty$. Using this theorem, the author is able to show the transcendence of the values of certain series, of which a special case is $\sum_0^\infty x^n/([c^n])!$, where $c>1$ and $x\neq 0$ is rational. *W. J. LeVeque.*

Pennisi, L. L. Elementary proof that e is irrational. Amer. Math. Monthly **60**, 474 (1953).

Djerasimović, Božidar. L'intervalle de l'erreur

$$\left|\sqrt[n]{c}-\frac{p}{q}\right|.$$

Bull. Soc. Math. Phys. Serbie **5**, no. 1–2, 53–56 (1953). (Serbo-Croatian. French summary)

ANALYSIS

Grebenyuk, D. G. On a method of interpolation. Akad. Nauk Uzbek SSR. Trudy Inst. Mat. Meh. **9**, 15–28 (1952). (Russian)

The author discusses the linear combination of n given functions which has the least deviation at $n+1$ given points from a given function, and compares it with the Lagrange interpolation "polynomial" (which agrees with the given function at n points). The paper consists mostly of numerical illustrations in one and two variables.

R. P. Boas, Jr. (Evanston, Ill.).

Inozemcev, O. I. On the theory of best approximation of functions of several variables by means of entire functions of finite degree. Doklady Akad. Nauk SSSR (N.S.) **91**, 15–18 (1953). (Russian)

The author sketches the proofs of two theorems (one direct, one inverse) connecting the order of differentiability of a function of n real variables with its degree of weighted approximation by entire functions of exponential type. The precise statements cannot be reproduced here, since about half the paper is devoted to establishing the terminology in which they are expressed. *R. P. Boas, Jr.*

Porcelli, Pasquale. Uniform completeness of sets of reciprocals of linear functions. Duke Math. J. **20**, 185–193 (1953).

The set K: $\{(1+k_px)^{-1}\}_{p=0}^\infty$ ($k_p\neq 0$, $k_p\neq k_q$ if $p\neq q$, and k_p non-ε $(-\infty, -1]$), is said to be uniformly complete in $F[0, 1]$ if each function continuous on $[0, 1]$ can be uniformly approximated by linear combinations of functions of K. Each of the following conditions is necessary and sufficient in order that K be uniformly complete in $F[0, 1]$: (i) the series $\sum_{p=0}^\infty(1-|x_p|)$ diverges, where

$$x_p=[(1+k_p)^{1/2}-1]/[(1+k_p)^{1/2}+1]$$

and $(1+k_p)^{1/2}$ denotes the value at $z=k_p$ of the branch of $(1+z)^{1/2}$ having at $z=0$ the value 1; (ii) the only solution $\varphi(x)$, complex-valued, normalized and of bounded variation on $[0, 1]$, of the system $\int_0^1(1+k_px)^{-1}d\varphi(x)=0$ $(p=0, 1, 2, \cdots)$ is $\varphi(x)\equiv 0$, $0\leqq x\leqq 1$; (iii) $M(K)$ contains 1, where $M(K)$ is the closed linear manifold whose elements are those continuous functions on $[0, 1]$ which are uniformly approximated by linear combinations of functions of K. In the more restrictive case where $|\arg(1+k_p)|\leqq\theta<\pi$, $|1+k_p|\geqq\delta>0$ $(p=0, 1, 2, \cdots)$ the divergence of $\sum_{p=0}^\infty|k_p|^{-1/2}$ is necessary and sufficient for uniform completeness of K in $F[0, 1]$. This latter result includes a result of van Herk [Compositio Math. **9**, 1–79 (1951); these Rev. **12**, 808]. Proofs involve a result of F. Riesz [Ann. Sci. Ecole Norm. Sup. (3) **28**, 33–62 (1911)] which includes (ii), and results of H. S. Wall [Analytic theory of continued fractions, Van Nostrand, New York, 1948, Ch. XV; these Rev. **10**, 32] on integral representations of certain classes of functions in the unit circle and in the plane cut from -1 to $-\infty$.

W. T. Scott (Evanston, Ill.).

Royden, H. L. Bounds on a distribution function when its first n moments are given. Ann. Math. Statistics **24**, 361–376 (1953).

The author is concerned with the reduced moment problem in the Stieltjes case, that is, given the moments M_k (about 0) of orders $k=0, 1, \cdots, n$ for a non-negative random variable X, to find exact bounds for its cumulative distribution function $F(X)$. He gives (Theorems 1 and 2) a self-contained derivation of the Čebyšev inequalities, which provide these bounds, and (Theorem 3) full details of various formulae required for the calculation of the bounds. He notes that the corresponding results for the Hamburger case, that is for a random variable in $(-\infty, \infty)$, are classical [cf. Shohat and Tamarkin, The problem of moments, Amer. Math. Soc. Math. Surveys, v. 1, New York, 1943, p. 43; these Rev. **5**, 5; **13**, 1138] and that his Theorems 1 and 2 are covered by more general results given by M. G. Kreĭn [Uspehi Matem. Nauk (N.S.) **6**, no. 4(44), 3–120 (1951), especially I, §4, pp. 46–53, and §10, pp. 86–91; these Rev. **13**, 445]. His paper provides, however, a convenient exposition for the case considered. He also obtains exact bounds for $F(X)$ when X has a unimodal distribution on $(-\infty, \infty)$ and its first two absolute moments about the mode are given. *H. P. Mulholland* (Birmingham).

Lalaguë, Pierre. Classes de fonctions indéfiniment dérivables presque périodiques de spectre donné. C. R. Acad. Sci. Paris **236**, 2473–2475 (1953).

Let $\{M_n\}$ be a sequence of positive numbers of which an infinite number are finite. Let E be a set of numbers $\lambda>0$ such that $\lambda\leqq\liminf M_n^{1/n}$. The author defines a E-regularization of the sequence $\{M_n\}$ denoted by $\{M_n^E\}$, and considers almost periodic infinitely differentiable functions $f(x)$ with spectrum E. Gorny type inequalities are announced of which the following is a typical example: If $|f^{(n)}(x)|\leqq M_n$, $n=1, 2, \cdots$, then $|f^{(n)}(x)|\leqq 6e^3nM_n^E$. The results are used to obtain inclusion theorems to classes of infinitely differentiable and almost periodic functions with spectrum E satisfying: (i) $|f^{(n)}(x)|\leqq c^nM_n$; (ii) $|f^{(n)}(x)|\leqq kM_n$.

S. Agmon (Jerusalem).

Bang, Thøger. The theory of metric spaces applied to infinitely differentiable functions. Math. Scand. 1, 137–152 (1953).

The author sets up a metric in the linear space of numerical sequences and applies it to the sequence $u_n(x)$ defined by $f^{(n)}(x)/(\epsilon^n m_n^\epsilon)$, where $m_n^{1/n} \to \infty$ and m_n^ϵ is the largest logarithmically convex minorant of m_n. By using properties of this metric he can give a real-variable proof of the sufficiency of the condition $\sum m_n^\epsilon / m_{n+1}^\epsilon = \infty$ for the quasianalyticity of the class of infinitely differentiable functions satisfying $|f^{(n)}(x)| \leqq m_n$ (this is the more difficult part of the Denjoy-Carleman theorem). He then gives a method (not a summation of the Taylor series) for calculating a quasianalytic function (i.e., a function satisfying the Denjoy-Carleman condition) from the values of its derivatives at a point. Next he considers a function with a sequence of zeros for its derivatives near (instead of at) a point, and shows that if $f(x)$ is quasianalytic and not identically zero, and $f^{(n)}(x_n) = 0$, then $\sum |x_{n-1} - x_n|$ diverges; as a corollary, he answers an old question of Borel's by showing that if all the derivatives of a quasianalytic function are positive at a point, the function is analytic. A more general form of the theorem includes, in particular, results on the location of the successive derivatives of various classes of analytic functions. Further theorems deal with the location of the zeros of a quasianalytic function itself [overlapping results of Hirschman, Amer. J. Math. 72, 396–406 (1950); these Rev. 11, 583] and with the number of zeros of $f^{(n)}(x)$ in a given interval. *R. P. Boas, Jr.* (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

***Fraenkel, Abraham A. Abstract set theory.** Studies in logic and the foundations of mathematics. North-Holland Publishing Co., Amsterdam, 1953. xii+479 pp. 38.00 florins.

This is an exposition, "chiefly intended for undergraduates in mathematics, graduate students in philosophy, and high school teachers", of the elementary classical theory of abstract sets (cardinal numbers, order types, and ordinal numbers). Attention is paid "to the foundations of the theory and to matters of principle in general, including points of logical or general philosophical interest." The author adopts a middle course between a naive and an axiomatic development of the theory: "certain principles, similar to Zermelo's axioms, are introduced and at the important turns of the exposition it is pointed out how the constructions required can be based on those principles, while elsewhere the development proceeds without explicit reference to the principles". The present volume is to be continued (by the same author) by another one (due to appear about 1955), "Foundations of set theory", dealing with the antinomies of the transfinite, axiomatic methods of founding set theory, logistic attitudes from "Principia mathematica" to the present day, intuitionism and neo-intuitionism, axiomatics in general and metamathematics. Some noteworthy features of the book under review are a correct treatment of definition by transfinite induction, two proofs of the comparability theorem (one based on order and the other on equivalence), and two proofs of the equivalence theorem (one relying on, and the other dispensing with, the sequence of natural numbers). Every section except the first is followed by a set of exercises, some of which are easy whereas others substantially extend the theory and are accompanied by references to the literature. At the end of the work there is an extensive, although not exhaustive, bibliography (128 pages) on the theory of abstract sets as well as its foundations. [It seems unfair to attach (p. 132) the name of Zermelo but not that of Jourdain to König's theorem. Of the infrequent misprints, that on p. 63, where "not" should be deleted from line 15 of the text, may trouble a beginner. Lusin's name is almost consistently misspelled "Łusin". Some awkward expressions are found here and there, such as (p. 128), "The cardinal of a well-ordered set is being called an aleph."]

F. Bagemihl (Princeton, N. J.).

Ohkuma, Tadashi. Structure of homogeneous chains. Kōdai Math. Sem. Rep. 1953, 1–12 (1953).

A homogeneous chain X is a simply ordered set in which for each x, y there is an automorphism φ of X such that $\varphi(x) = y$. A regular interval of X is an interval $Y \subseteq X$ such that for each automorphism φ of X such that $\varphi(Y)$ meets Y it follows that $\varphi(Y) = Y$. Given a proper interval I of X the definition $x \sim y$ if $\varphi(x) \in Y$ if and only if $\varphi(y) \in Y$ yields a decomposition of X into disjoint regular intervals; the decomposition is called a regular division of X; each regular division of α of X is itself a chain.

A chain X is called simple if there are no proper regular intervals. Theorems 1 to 6 describe the structure of simple homogeneous chains. If X is not simple, consider the family of regular divisions of X; it is shown that of two such divisions one is a refinement of the other; the index chain A of X is the subchain consisting of those regular divisions α of X such that α has an immediate refinement α', that is, an immediate successor in the chain of regular divisions. If $a \in \alpha \in A$, the chain of those elements of α' which are contained in a is a simple homogeneous chain X_a determined up to isomorphism by α. For each α, X is isomorphic to the ordinal product $\alpha \circ X_a$; X is also isomorphic to a subchain of a lexicographic product $\prod_{a \in A} X_a$, [defined here as in Ohkuma, same Rep. 1952, 23–30; these Rev. 13, 828]. It is not determined whether equality must hold for every X, but it is observed that this problem is related to the question of whether a non-simple X has a minimal proper regular interval. *M. M. Day* (Urbana, Ill.).

Enomoto, Shizu. Boolean algebras and fields of sets. Osaka Math. J. 5, 99–115 (1953).

The author applies the concept of ramification set, due to Horn and Tarski [Trans. Amer. Math. Soc. 64, 467–497 (1948); these Rev. 10, 518], to the construction of isomorphisms between various kinds of Boolean algebras and corresponding types of fields of sets. A typical theorem is the following. For a Boolean algebra A the following conditions are equivalent. (1) A is isomorphic with a field of sets which is completely additive in the wider sense. (2) For every a in A which is not zero there exists a two-valued measure which is strongly completely additive in the wider sense and which takes the value 1 at a. (3) A is atomic. (4) A is completely distributive in the wider sense. [It seems to the reviewer that most of the author's results do not need the complicated constructions involving ramification sets, but can be derived simply by considering appropriate dense subsets of the Stone representation space.]

L. H. Loomis (Cambridge, Mass.).

Neumer, Walter. Bemerkungen zur allgemeinen Hypothese $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ im Anschluss an einige Beweisversuche von H. Eyraud. Math. Nachr. 9, 321–342 (1953).

This is a detailed analysis of concepts and methods employed by Eyraud [cf. Leçons sur la théorie des ensembles, les nombres transfinis, et le problème du continu, 2d ed., Institut de Mathématiques, Lyon, 1949; II (supplement), 1950; these Rev. 12, 166; 13, 120] in unsuccessful attempts to solve the continuum problem. The author presents rigorous versions as well as extensions of Eyraud's definitions and correct results, and points out fallacies in Eyraud's arguments purporting to establish the validity of the continuum hypothesis. This hypothesis is shown to be equivalent to each one of two other propositions. Let λ be a limit number. An ascending λ-line is a sequence of ordinal numbers $\{\alpha_\xi\}_{\xi<\lambda}$ such that $\alpha_\mu<\alpha_\nu<\lambda$ for $\mu<\nu<\lambda$. If $A=\{\alpha_\xi\}_{\xi<\lambda}$ and $B=\{\beta_\xi\}_{\xi<\lambda}$ are ascending λ-lines, and if there exists an $\iota<\lambda$ such that $\alpha_\xi<\beta_\xi$ for $\iota\leqq\xi<\lambda$, then we write $A\dashv B$. A set $\mathfrak{N}$ of ascending λ-lines which is well-ordered and of type τ with respect to the relation $\dashv$ is called a network of length τ. A prime line $P\in\mathfrak{N}$ is a line possessing an initial segment which is not an initial segment of any line $Q\in\mathfrak{N}$ with $Q\dashv P$. Let $0\subset E'\subseteq E$, where E is a well-ordered set. We say that a regressive function f of order δ is defined in E if, for every $a\in E'$, $f(a)$ is a nonempty subset of E such that every element of $f(a)$ precedes a, $f(a)=f(a')$ if and only if $a=a'$ $(a, a'\in E')$ and δ is the smallest ordinal such that $|f(a)|<\aleph_\delta$ for every $a\in E'$. Consider the following propositions: (K_α) $2^{\aleph_\alpha}=\aleph_{\alpha+1}$. (P_α) The number of prime lines of a network, of length $\omega_{\alpha+2}$, of ascending $\omega_{\alpha+1}$-lines is $\aleph_{\alpha+1}$. (R_α) If a regressive function f of order $\alpha+1$ is defined in E, and $|E-E'|\leqq\aleph_{\alpha+1}$, then $|E|\leqq\aleph_{\alpha+1}$. For every ordinal α, (K_α) is equivalent to (P_α) as well as to (R_α). [Theorem 16 is not new; it is proved in Tarski, Fund. Math. 16, 181–304 (1930), p. 195, who refers to Sierpiński, Zarys teorji mnogości [Foundation of the theory of aggregates], part 1, 3d ed., Warszawa, 1928, pp. 251–253.] As the author remarks, his paper has some points of contact with a paper by Cuesta [Revista Mat. Hisp.-Amer. (4) 9, 83–96, 168 (1949); these Rev. 11, 646]. *F. Bagemihl* (Princeton, N. J.).

Fodor, G. An assertion which is equivalent to the generalized continuum hypothesis. Acta Sci. Math. Szeged 15, 77–78 (1953).

For every ordinal number α, the following two propositions are equivalent: (1) $2^{\aleph_\alpha}=\aleph_{\alpha+1}$. (2) Let $|E|=2^{\aleph_\alpha}$, and denote by B the class of all subsets of E of power 2. Then there exists a mapping, T, of B into E such that (a) if $r=\{x, y\}\in B$, then either $T(r)=x$ or $T(r)=y$, and (b) if $E_1\subseteq E$ and $|E_1|>\aleph_\alpha$, then E is equal to the union of the sets $r\in B$ for which $T(r)\in E_1$. *F. Bagemihl.*

Chopoff, Peter P. Classes de Baire. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie I. 47, 157–181 (1951). (Bulgarian. French summary)

This is a treatise on the theorem of Baire and the problem of Baire for functions of Baire class $\alpha<\Omega$ [see de la Vallée Poussin, Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire, 2nd ed., Gauthier-Villars, Paris, 1934, pp. 113–166]. The author claims to avoid the use of transfinite methods by introducing notions of constructive sets and functions and dealing only with them. However, it is the reviewer's opinion that the term "constructive sequence" is inadequately defined and that the presentation offered differs in no essential respect from that given by de la Vallée Poussin [loc. cit.]. *E. Hewitt.*

Pauc, Christian. Ableitungsbasen, Prätopologie und starker Vitalischer Satz. J. Reine Angew. Math. 191, 69–91 (1953).

This paper is a detailed exposition of the results announced by the author in two notes [C. R. Acad. Sci. Paris 234, 1119–1120, 1242–1243 (1952); these Rev. 13, 829].

T. H. Hildebrandt (Ann Arbor, Mich.).

Kappos, Demetrios A. Erweiterung von Massverbänden. J. Reine Angew. Math. 191, 97–109 (1953).

The main result is the existence of a σ-extension of a measure algebra. In other words, starting with a Boolean algebra and a strictly positive, normalized, finitely additive measure, the author constructs a larger algebra and an extension measure that is countably additive. Under suitable minimality conditions, the extension is unique. The methods are mainly metric: The extension algebra is constructed as a set of Cauchy sequences modulo null sequences.

P. R. Halmos (Chicago, Ill.).

Karlin, Samuel. Extreme points of vector functions. Proc. Amer. Math. Soc. 4, 603–610 (1953).

Let μ be a finite measure on a Borel field F of sets of points of an abstract space X. Let $M(\mu, F)$ be the space of all essentially bounded measurable functions on X. Let $(\otimes M^n)$ be the direct product of M with itself n times. Let A be a bounded closed convex set in Euclidean n-space. Let B be the set of extreme points of A, and $\bar{B}$ be the closure of B. Let M_A be the set of all x in $(\otimes M^n)$ whose range of values lies almost everywhere in A. Theorem 1: The extreme points of M_A are contained in $M_{\bar{B}}$. Theorem 2: Let μ_i, $i=1, \cdots, n$, be atomless finite measures on F. The extreme points of the set T of all measurable functions x such that $x\in M_A$ and $b_i\leqq\int x d\mu_i\leqq a_i$ are contained in $M_{\bar{B}}$. Theorem 3: Let $M[A, \mu]=\{\int x d\mu_i\}$, $i=1, \cdots, n$, x in M_A. Let μ_i, $i=1, \cdots, n$, be atomless, finite measures. Then

$$M[A, \mu]=M[\bar{B}, \mu].$$

Theorem 4: If $B=\bar{B}$, the set of extreme points of M_A is not weak * closed and the extreme points of M_B and M_A are sequentially weakly closed. Theorem 5: The extreme points of the set L of all positive regular measures over a topological space X satisfying $\int d\mu(x)=1$, $\int h_i(x)d\mu=a_i$, $i=1, \cdots, n$, with $h_i(x)$ continuous, consist of those measures in L with not more than $n+1$ points of increase provided that the rank of $\{h_i(x_j)\}$ is $n+1$ where $i, j=0, 1, \cdots, n$, $h_0(x)\equiv 1$, and x_j are distinct points of X. *J. Wolfowitz.*

Henstock, R., and Macbeath, A. M. On the measure of sum-sets. I. The theorems of Brunn, Minkowski, and Lusternik. Proc. London Math. Soc. (3) 3, 182–194 (1953).

Denote by A, B sets in n-dimensional Euclidean space. A' denotes the complement of A; $H(A)$ denotes the convex closure of A; $A+B$ denotes the set $\{x+y\}$, $x\in A$, $y\in B$, where $x+y$ means vector addition. Lebesgue n-dimensional measure, inner measure, and outer measure are denoted by μ, μ_*, μ^*, respectively. $\Lambda(A)=(\mu(A))^{1/n}$. The authors prove among others the following theorems: Theorem 1. If A and B are measurable and nonempty, then (1) $\Lambda_*(A+B)\geqq\Lambda(A)+\Lambda(B)$. Theorem 2. If $0<\mu(A)\mu(B)<\infty$ and equality holds in (1), then

$$\mu(H(A)\cap H')=\mu(H(B)\cap B')=0, \tag{2}$$

(3) $H(A)$ and $H(B)$ are homothetic. Conversely, if $0<\mu(A)\mu(B)<\infty$ and (2) and (3) hold, we have equality

in (1). Theorem 3. If $\mu(A)=0$ and $0<\mu(B)<\infty$, then the necessary and sufficient condition for equality in (1) is that A should consist of one point only. If $\mu(A)=\mu(B)=0$, it does not seem easy to give a necessary and sufficient condition for $\mu(A+B)=0$. Theorem 1 was proved for convex sets by Brunn and conditions for equality were found by Minkowski. Theorems 1, 2, and 3 were stated by Lusternik [C. R. (Doklady) Acad. Sci. URSS (N.S.) 3, 55–58 (1935)] but his proofs of theorems 2 and 3 are stated by the authors to be defective. *P. Erdös* (South Bend, Ind.).

Macbeath, A. M. On measure of sum sets. II. The sum-theorem for the torus. Proc. Cambridge Philos. Soc. 49, 40–43 (1953).

Let A and B be subsets of the r-dimensional torus whose points x are r-tuples $(x_1, x_2, \cdots, x_r)$ where the x_i are residue classes mod 1. $A+B$ is defined as the set of all points $x+y$, $x \in A$, $y \in B$. The author proves the following theorem: Let A and B be measurable non-empty sets. Then

$$\mu_*(A+B) \geqq \min(1, \mu(A)+\mu(B)), \tag{1}$$

where μ is the Haar measure of the r-dimensional torus and μ_* is the inner measure. The author makes a plausible conjecture when equality holds in (1). *P. Erdös.*

***Cafiero, Federico. Funzioni additive d'insieme ed integrazione negli spazi astratti.** Libreria Editrice Liguori, Napoli, 1953. 178 pp.

The general presentation and the notation, which some mathematicians now consider antiquated, do not differ greatly from that of the first chapter of Saks, Theory of the integral [Warszawa-Lwów, 1937]. The material has been made to include an abstract reformulation of uniform absolute continuity [Doubrovsky, Mat. Sbornik N.S. 20(62), 317–329 (1947); these Rev. 9, 19] and the author's formulation of a corresponding theorem of Vitali. The proof of the well-known theorem of Nikodým has been recast. The author has included also an account of the theorem of Banach-Kuratowski according to which, subject to the continuum hypothesis, a completely additive set function defined for all subsets of a space of the potency of the continuum is identically zero if it vanishes for finite sets.

There is one inclusion which the reviewer cannot understand and which deals with an extension theorem reproduced from papers and lectures by Caccioppoli. This extension theorem appears to be open to an objection pointed out by Radó and Reichelderfer [see Radó, Length and area, Amer. Math. Soc. Colloq. Publ., v. 30, New York, 1948, p. 421; these Rev. 9, 505]. Its various forms occur in several places in the author's text, and its meaning is obscured by an initial misprint [in condition α) on p. 15]. *L. C. Young.*

Cafiero, Federico. Sul passaggio al limite sotto il segno d'integrale di Stieltjes-Lebesgue negli spazi astratti, con masse variabili con gli integrandi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 488–494 (1953).

Consider a sequence φ_n of completely additive (ca) set-functions defined on a σ-field F of subsets of S, and a sequence f_n of F-measurable functions on S. Suppose that $\varphi_n(e) \to \varphi(e)$ for each $e \in F$ and that $f_n(x) \to f(x)$ for each $x \in S$. Nikodým [Monatsh. Math. Phys. 40, 427–432 (1933)] showed φ completely additive and that, if f_1 is also bounded, $\int f_1 d\varphi_n \to \int f_1 d\varphi$.

A family Φ of ca set-functions is called uniformly additive on F if for each decreasing sequence $\{I_n\} \subseteq F$ with empty intersection and for each $\epsilon>0$ there is $n(\epsilon)$ such that, for each $\varphi \in \Phi$, $|\Phi(I_n)|<\epsilon$ if $n>n(\epsilon)$. For ca functions this is equivalent to equi-absolute continuity with respect to some σ-finite measure on F [Caccioppoli, Rend. Accad. Sci. Fis. Mat. Napoli (3) 35, 34–40 (1929); Dubrovskiĭ, Doklady Akad. Nauk SSSR (N.S.) 58, 737–740 (1947); these Rev. 9, 275]. Nikodým also showed that $\psi_n(e)=\int_e f_n d\varphi_n$ uniformly additive sufficed for $\psi_n(e) \to 0$ when the f_n are summable with respect to the corresponding φ_n and $f=0$. This note reviews the above relations and then shows: If f_n are summable with respect to φ_n, then $\psi_n(e)$ tends to $\int_e f d\varphi$ for each $e \in F$ if and only if the functions ψ_n are uniformly additive on F. *M. M. Day* (Urbana, Ill.).

Erim, Kerim. Stieltjessche Integrale. Rend. Circ. Mat. Palermo (2) 1 (1952), 332–342 (1953).

The author extends to the case of n dimensions the definition of the Stieltjes integral which was formulated by the reviewer [Bull. Amer. Math. Soc. 43, 581–588 (1937)]. The integral is proportional to the Cesàro mean of the values of the integrand at an infinite sequence of points. The sequence is determined (but not uniquely) by the function with respect to which one is integrating (assumed non-decreasing). Integration with respect to an arbitrary function of limited variation is the difference of the integrals with respect to the positive and negative variations. The author obtains a new method of constructing the sequence of points. *A. H. Copeland, Sr.*

Schubert, Horst. Zur Differenzierbarkeit der Abbildungen von Punktmengen des euklidischen Raumes. J. Reine Angew. Math. 191, 1–12 (1953).

Let $f(P)$ be a function defined on a point set B in euclidean n-space R^n. The problem is to define a differential $df=\nabla f(P_0)\cdot dP$ that shall be uniquely determined at all points P_0 of B and shall meet certain natural requirements. Now $dP=P-P_0$ where P may range, for the purpose of defining df, either (a) over all of R^n or (b) over the linear space at P_0 spanned by the half-tangents to B at P_0. While the results are the same if P_0 is an interior point of B, they differ at a frontier point. The theory in case (a), developed here with extensive use of the graph $(P, f(P))$ in $R^n \times R^1$, avoids difficulties that arise in case (b) in defining higher derivatives and in dealing with the affine connection when B is a Riemannian manifold imbedded in R^n.

F. A. Ficken (Knoxville, Tenn.).

Babič, V. M. On the extension of functions. Uspehi Matem. Nauk (N.S.) 8, no. 2(54), 111–113 (1953). (Russian)

Let A be a closed bounded region in E_n whose boundary is piecewise of class $C^{(k)}$, and let $f(p)$, $p \in A$, be a function defined in A and of class $C^{(k)}$ in A, i.e., f is continuous with all its partial derivatives of order k in A. H. Whitney [Trans. Amer. Math. Soc. 36, 63–89 (1934)] and M. R. Hestenes [Duke Math. J. 8, 183–192 (1941); these Rev. 2, 219] have proved various theorems on the extension of functions, in particular that every function of class $C^{(k)}$ in A has an extension which is of class $C^{(k)}$ in E_n. The author says that a function f is of class $W_p^{(k)}$ if the continuity requirement of the partial derivatives of order k in the definition of class $C^{(k)}$ is replaced by L^p-integrability. Then he proves that every function f of class $W_p^{(k)}$ in A has an extension which is of class $W_p^{(k)}$ in E_n. Hestenes' method based on a modified form of reflection principle is used. *L. Cesari.*

***Császár, Ákos. Sur la propriété de Darboux.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 551–560. Akadémiai Kiadó, Budapest, 1952. (Hungarian and French. Russian summary)

Let $f(x)$ be real-valued on (a, b). For each x_0 let $I_r(x_0)$ denote the open interval whose end-points are lim inf and lim sup of $[f(x); x \geqq x_0, x \to x_0]$; let $I_l(x_0)$ be defined similarly, using $x \leqq x_0$, $x \to x_0$. Then f is called "Darboux at x_0" if for each $h>0$, the values assumed by $f(x)$ on $x_0<x<x_0+h$ cover (i.e., include) $I_r(x_0)$ and the values assumed by $f(x)$ on $x_0-h<x<x_0$ cover $I_l(x_0)$ (only the first condition to be used if x_0 is the end-point a and only the second condition to be used if x_0 is b). The author proves: Theorem 1. f has the Darboux property on (a, b) if and only if it is Darboux at each point of the closed interval (a, b). Theorem 2. The same result holds if, in both hypothesis and conclusion, the word cover means "includes, with the possible exception of an N-set". Here the N-sets can be any family of sets with the properties: (a) every single point is an N-set; (b) every subset of an N-set is again an N-set; (c) the set-union of a countable collection of N-sets is an N-set. Theorem 2 does not include Theorem 1 because of condition (a). [The reviewer notes that a simpler proof shows that Theorem 2 is valid when (a) is weakened to (a'): If an open interval is an N-set then its closure is also an N-set; which includes Theorem 1 where the only N-set is the empty set.]

I. Halperin (Kingston, Ont.).

***Gehér, István. Sur une transformation d'intégrale.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 507–518. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and French summaries)

While in classical analysis the transformation formula for definite integrals is considered only for the case when the change to new variables represents a one-to-one transformation, this requirement has been found essentially irrelevant in modern researches in this field. Furthermore, the classical requirements concerning "smoothness" were replaced by assumptions based on the concept of absolute continuity [see for example the reviewer's book, Length and area, Amer. Math. Soc. Colloq. Publ., vol. 30, New York, 1948, part IV; these Rev. 9, 505]. The present paper contains various interesting contributions to this line of thought. The following theorem is typical of the type of results obtained. Let $f(x, y)$ be a continuous function in the unit square Q: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. On the z-axis in (x, y, z)-space, let H be a Borel measurable set and $\varphi(z)$ a Borel measurable function on H. For each real number z, denote by $m(z)$ the exterior linear measure of the set of those points (x, y) where $f(x, y)=z$. The function $\varphi(f(x, y))$ of (x, y) gives rise to the set $E=\varphi^{-1}(H) \subset Q$. Assume that $f(x, y)$ is absolutely continuous in Q in the Tonelli sense. Then

$$\int_H \varphi(z)m(z)dz = \int\int_E \varphi(f(x, y))(f_x^2+f_y^2)^{\frac{1}{2}}dxdy,$$

as soon as one of the two integrals involved exists.

T. Radó (Columbus, Ohio).

Cecconi, Jaurès. Sulla additività ciclica degli integrali sopra una superficie. Rivista Mat. Univ. Parma 4, 43–67 (1953).

Let $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ be a representation of an oriented Fréchet surface of finite Lebesque area, where $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$. Consider a function $f(x, y, z, u, v, w)$, which is defined and continuous for all arguments x, y, z, u, v, w such that $u^2+v^2+w^2>0$ and (x, y, z) is a point of S, and is positively homogeneous of degree one with respect to u, v, w. The purpose of the paper is to extend the so-called cyclic additivity theorem to the integral (in the sense of Cesari) $\int f da$ taken over S. Let S_n, $n=1, 2, \cdots$, be the oriented Fréchet surfaces obtained by retractions upon the proper cyclic elements of the middle-space occurring in the monotone-light factorization of the given representation of S. Let I be the Cesari integral of f over S, and let I_n have the same meaning relative to S_n. If L, L_n are the Lebesgue areas of S, S_n, then one has $L=\sum L_n$, the cyclic additivity theorem for Lebesgue area [see the reviewer's book, Length and area, Amer. Math. Soc. Colloq. Publ., vol. 30, New York, 1948; these Rev. 9, 505]. In the present paper, it is shown that $I=\sum I_n$, a fundamental result which may be termed the cyclic additivity theorem for Cesari integrals. The reviewer takes the liberty of calling attention to the following situation. In view of the most gratifying progress accomplished in recent years in the theory of double integrals over Fréchet surfaces, it is natural to turn to the corresponding problems for n-tuple integrals. In the theory of double integrals, the so-called middle-space topology is fundamental (see, for example, the reviewer's book cited above). Several workers in this field have commented upon the regrettable fact that a large proportion of the results in this type of topology ceases to be valid in dimensions $n>2$. Accordingly, the study of n-tuple integrals (in the direction represented by the paper under review) may be expected to disclose fruitful topological problems of a novel character.

T. Radó (Columbus, Ohio).

Cecconi, Jaurès. Sull'approssimazione delle superficie di Fréchet. Rivista Mat. Univ. Parma 4, 69–82 (1953).

Let (T, Q): $p=p(w)$, $w \in Q$, denote any continuous mapping from the closed unit square Q of the w-plane E_2 into the Euclidean p-space E_3, and let (T, Q^*) denote the mapping from the boundary Q^* of Q into E_3 defined by T. Then (T, Q) determines a continuous parametric Fréchet surface S, and (T, Q^*) a continuous parametric closed Fréchet curve which is denoted by θS, or boundary curve of S. Let $\|S, S'\|$ denote the Fréchet distance of two surfaces S, S', and $L(S)$ the Lebesgue area of S. Then S is called a polyhedral surface if S has a representation, say (T, Q), which is quasi-linear in Q, i.e., there is a finite subdivision D of Q into triangles t such that T is linear on each $t \in D$; S is called a quasi-polyhedral surface if S has a representation, say (T, Q), which is quasi-linear on each closed polygonal region R interior to Q. In both cases the elementary area $a(S)$ of S can be defined as usual. The author proves the following main theorem: For any surface S of finite area $L(S)$ and for any $\epsilon>0$ there exists a quasi-polyhedral surface S with $\theta S'=\theta S$, $\|S, S'\|<\epsilon$, $|a(S')-L(S)|<\epsilon$. The present theorem is of interest in calculus of variations and particularly in surface area theory. Applications are announced. Previous results of Mambriani [Ann. Scuola Norm. Super. Pisa (2) 13, 1–17 (1948); these Rev. 9, 417] and the reviewer [Rivista Mat. Univ. Parma 1, 207–227 (1950); these Rev. 12, 86] are used in the proof.

L. Cesari (Lafayette, Ind.).

Cecconi, Jaurès. Un complemento al teorema di Stokes. Rend. Sem. Mat. Univ. Padova **22**, 23–37 (1953).

Let $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$ be a representation of an oriented Fréchet surface S, where $0\leq u\leq 1$, $0\leq v\leq 1$. Assume that the Lebesgue area of S is finite and the boundary curve of S is rectifiable. Let $F(x, y, z)$ be a function such that F and its partial derivatives F_y and F_z are continuous in an open set R containing S. Using Weierstrass-type integrals in the sense of Tonelli and Cesari, the author established previously the Stokes integral identity under the conditions just stated. In the present paper it is shown that the assumption of continuity of F, F_y, F_z can be replaced by the assumption of approximate continuity. The method is based largely upon concepts and results developed in the modern theory of surface area. *T. Radó.*

Cecconi, Jaurès. Sul teorema di Gauss-Green per una particolare classe di superficie. Rend. Sem. Mat. Univ. Padova **22**, 81–112 (1953).

This paper is an important contribution to the difficult problem of determining the range of validity of the Gauss-Green integral identity for closed surfaces in 3-space. While, in general, it must be assumed that the points of the given closed surface S constitute a set of (three-dimensional) measure zero, it has been discovered earlier that this restriction can be dispensed with if S is a Jordan surface. For this important special case, the author improves previous results in the following direction. A basic assumption concerning S may be expressed in terms of the concept of bounded variation, as applied to the projection of S upon a coordinate plane (this projection being considered as a transformation from the parameter region into that coordinate plane). The author uses a very general concept of bounded variation, developed by several workers independently [the concept which is termed essential bounded variation in the reviewer's book, Length and area, Amer. Math. Soc. Colloq. Publ., vol. 30, New York, 1948; these Rev. **9**, 505]. Accordingly, he obtains a result of impressive and gratifying generality concerning the validity of the Gauss-Green identity for the case of closed Jordan surfaces. It is a matter of regret to the reviewer that the diversity and complexity of the basic definitions involved makes it impracticable to give here a detailed account of the contents of this significant paper. *T. Radó* (Columbus, Ohio).

***Cecconi, Jaurès. Una osservazione sul teorema di Gauss-Green.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 50–51. Casa Editrice Perrella, Roma, 1953.

The range of validity of the Gauss-Green integral identity, for closed surfaces of the type of the 2-sphere in 3-space, has been studied by many workers. In the present note, the author exhibits an example to the effect that the identity does not generally hold if the points of the surface constitute a set of positive three-dimensional measure. *T. Radó.*

Theory of Functions of Complex Variables

***Goluzin, G. M. Geometričeskaya teoriya funkcil kompleksnogo peremennogo. [Geometrical theory of functions of a complex variable.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 540 pp. 16.85 rubles.

This text is devoted to advanced topics in the theory of functions of a complex variable, and as the title indicates the emphasis is on the geometric parts of the theory.

The number of topics touched is so large that a detailed account is impossible in a review. Most of the material however has already appeared in the literature, and the authors chief contribution is in organizing and simplifying the material and making it readily accessible. The book is written in a clear and attractive style, due attention is given to motivation, and the selections from the literature were well made. The reviewer noted only two omissions: (1) No mention is made of multivalent functions and (2) the author did not include his own necessary and sufficient conditions on the coefficients of a power series in order that the function be univalent [Doklady Akad. Nauk SSSR (N.S.) **81**, 721–723 (1951); these Rev. **13**, 546].

A. W. Goodman (Lexington, Ky.).

Shniad, H. Convexity properties of integral means of analytic functions. Pacific J. Math. **3**, 657–666 (1953).

Let $f(z)$ be regular in $|z|<1$. It is shown that the mean $M_4(r; f)=[(2\pi)^{-1}\int_0^{2\pi}|f(re^{it})|^4dt]^{1/4}$, of order 4, is a convex function of r for all $0\leq r<1$. On the other hand, there exists a function $f(z)$ for which the mean $M_t(f; r)$ is not convex when $t>5.66$ [cf. Beckenbach, Gustin, and Shniad, Bull. Amer. Math. Soc. **55**, 184–190 (1949); these Rev. **10**, 441].

W. W. Rogosinski (Newcastle-on-Tyne).

San Juan, Ricardo. Analysis of two series studied by Hardy. Revista Mat. Hisp.-Amer. (4) **13**, 128–145 (1953). (Spanish. French summary)

The author analyses carefully two examples given by Hardy [Messenger of Math. **43**, 22–24 (1913); Divergent series, Oxford, 1949; these Rev. **11**, 25] in each of which the Borel sum of a certain series represents two different functions in the adjacent regions in which it is valid. Hardy showed that each of the functions is the analytic prolongation of the other in the respective regions. In the first example the series is $\sum_0^\infty(-1)^n(2n)!/(n!z^{2n})$ and the author proves that it has the normal radial sum

$$f_1(z)\equiv ze^{z^2/4}\left[\tfrac{1}{2}\pi^{1/2}-\int_0^{z/2}e^{-v^2}dv\right]$$

in $\Re(z)>0$ and $f_2(z)\equiv f_1(z)-\pi^{1/2}ze^{z^2/4}$ in $\Re(z)<0$. Inside the respective half-planes $f_1(z)$, $f_2(z)$ approximate asymptotically to the series and these are the best approximations and radial prolongations [Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. **2**, 185–195 (1952); these Rev. **14**, 1078] in these regions. The analytical prolongation of either into the field of the other merely gives an asymptotic approximation which can be obtained from the best approximation by the addition of a function having a zero asymptotic approximation in that region. Applications are given to Fresnel integrals which can be developed into similar series.

The second example concerns the series

$$\sum_{n=0}^{\infty}(-1)^n z^{-n}\left(\sum_{v=0}^{\infty}(-1)^v v^n/v!\right),$$

and although it gives rise to a similar result, it is more difficult to handle. In the first example the Borel sum is shown to coincide with a Stieltjes integral having a positive decreasing kernel, whereas in the second example the kernel is a non-monotonic increasing step-function.

There are a number of misprints and minor errors.

R. Wilson (Swansea).

Azpeitia, A. G. On a generalization of the Fresnel integrals. Revista Mat. Hisp.-Amer. (4) **13**, 61–80 (1953). (Spanish. French summary)

In the paper reviewed above San Juan has obtained expressions for the Fresnel integrals both as a Cauchy series

and in the Stieltjes form. The author generalises this work by considering the functions $\int_0^s \cos \frac{1}{2}\pi z^{2k} dz$, $\int_0^s \sin \frac{1}{2}\pi z^{2k} dz$, and expressing them in terms of

$$F_{2k,\lambda}(z) = z^{-1} \exp\left(z^{-2k} + i\frac{\lambda}{\pi} - k\right)\Gamma\left(\frac{1}{2k}\right) - \int_0^{1/z} \exp(-s^{2k})ds$$

which coincides in the sector

$$(2\lambda - 1)\pi/2k < \arg z < (2\lambda + 1)\pi/2k$$

with a generalised form of the first of Hardy's series studied in San Juan's paper. The $F_{2k,\lambda}(z)$, and through them the integrals, are finally expressed as generalised Cauchy series and their radial prolongations, and also in terms of Stieltjes integrals. The minor errors and misprints make parts of the paper difficult to follow. *R. Wilson* (Swansea).

Delange, Hubert. Sur un théorème de Pólya. Bull. Sci. Math. (2) 77, 56–62 (1953).

The author gives a new proof to a particular case of a well known theorem of Pólya relating the density of the sequence of exponents of a Dirichlet series and the maximal length of a singularity-free segment on the axis of convergence. *S. Agmon* (Jerusalem).

Rajagopal, C. T. A note on power series. Math. Student 20 (1952), 99–106 (1953).

The principal contributions of this paper are analogues to familiar theorems of Rogosinski, Steffensen, Bohr, and Landau on functions $f(z) = \sum a_n z^n$ which are regular and bounded in the unit disc [see Landau, Darstellung und Begründung . . . , 2. Aufl., Springer, Berlin, 1929, pp. 9, 10, 26]. Let the hypothesis that $f(z)$ is bounded in $|z| < 1$ be discarded, and instead suppose merely that 1 is an upper bound for the real part of $f(z)$ in $|z| < 1$. The author shows that then (1) $\Re s_n(z) \leqq 1$ in $|z| \leqq 1/2$ for $n = 0, 1, \cdots$, where $s_n(z) = \sum_0^n a_k z^k$; (2) $\Re \sigma_n(z) \leqq 1$ in $|z| \leqq 1$, where $\sigma_n(z) = (n+1)^{-1} \sum_0^n s_k(z)$; and (3) if also $a_0 \geqq 0$, then $M(1/3) \leqq 1$ and $M(r) \leqq 2/(1-r)$, where $M(r) = \sum |a_n| r^n$. The example $f(z) = 2z/(1+z)$ shows that the constants 1/2 and 1/3 in (1) and (3) cannot be increased, and that the coefficient 2 in (3) cannot be replaced by a smaller constant. Strengthening his hypothesis on $f(z)$ by requiring that $|\Re f(z)| \leqq 1$ in $|z| < 1$, the author obtains (for $|z| < 1$) the estimate

$$|\Re s_n(z)| \leqq L_n \sim 4\pi^{-2} \log n;$$

and for each n he constructs a function $f(z)$ for which $s_n(1) = L_n$. *G. Piranian* (Ann Arbor, Mich.).

Gaier, Dieter. Complex Tauberian theorems for power series. Trans. Amer. Math. Soc. 75, 48–68 (1953).

Les résultats indiqués dans ce mémoire constituent la thèse de l'auteur [Univ. of Rochester, 1951]. Voici les principaux. Soit $z = |z| e^{i\phi}$. Les domaines D_1, D_2, D_3, D_4 sont définis par: (1) D_1: $|z| < R$, $R > 1$, $|\arg(z-1)| > \theta_0$, $0 < \theta_0 < \pi/2$; (2) D_2: $|z-1| < \delta_0$, $\delta_0 > 0$, $|\arg(z-1)| > \theta_0$, $0 < \theta_0 < \pi/2$; (3) D_3: $|z| < R$, $R > 1$, $|z| < 1 + c_0\phi^2$, $c_0 > 0$; (4) D_4: $|z-1| < \delta_0$, $\delta_0 > 0$; $|z| < 1 + c_0\phi^2$, $c_0 > 0$. L'auteur désigne par V l'une des méthodes de sommation E_p (Euler), B (Borel), S_α où les coefficients de $S_k - u_0 + \cdots + u_k$ sont respectivement $2^{-p-n} C_n^k (2^p - 1)^{n-k}$ $(k = 0, 1, \cdots, n, p > 0)$, $e^{-x} x^k/k!$, $(1-\alpha)^{n+1} C_{n+k}^n \alpha^k$ $(n, k = 0, 1, \cdots, 0 < \alpha < 1)$. A désigne la méthode d'Abel. 1) Si $\sum a_n z^n$ de rayon de convergence 1 est régulière et bornée dans D_2 et si $a_n = o(1)$, si $\sum a_n$ est sommable (A) et a pour somme S, alors $\sum a_n = S$. (Généralisation d'un théorème de Fatou [Acta Math. 30, 335–400 (1906)] et d'un théorème de M. Riesz où on utilise D_1 [Landau, Darstellung und Begründung einiger neueren Ergebnisse der Funktionentheorie, 2. Aufl., Springer, Berlin, 1929, p. 64].) 2) Si $\sum a_n z^n$ de rayon de convergence 1 est régulière dans D_3, continue dans $\bar{D}_3$ et si $\sum a_n$ est sommable par V, alors $\sum a_n$ converge. 3) Si $\sum a_n z^n$ de rayon de convergence 1 est régulière et bornée dans D_4, si $a_n = o(1)$ et si $\sum a_n$ est sommable par V, $\sum a_n$ converge. Les autres résultats concernent d'une part la sommabilité par E_p ou par S_α de $\sum a_n$, d'autre part la sommabilité par B.

M. Zamansky (Paris).

Gaier, Dieter. Schlichte Potenzreihen an der Konvergenzgrenze. Math. Z. 58, 456–458 (1953).

By means of an analytic construction, Herzog and the reviewer have exhibited a function which is regular, bounded, and schlicht in $|z| < 1$, and whose Taylor series converges everywhere on the unit circle C but not uniformly on any arc of C [Duke Math. J. 20, 41–54 (1953); these Rev. 14, 738]. The author points out that if the function $w = \sum a_n z^n$ maps the unit disc $|z| < 1$ conformally upon a bounded region G whose prime ends are all of the first or second kind, then $\sum a_n z^n$ converges everywhere on C. If in addition every arc of prime ends of G contains prime ends of the second kind (as is the case with a certain region constructed by Carathéodory [Math. Ann. 73, 323–370 (1913), see p. 366]), then the convergence of $\sum a_n z^n$ is not uniform on any arc of C. *G. Piranian* (Ann Arbor, Mich.).

Thron, W. J. A class of meromorphic functions having the unit circle as a natural boundary. Duke Math. J. 20, 195–198 (1953).

The C-fraction $1 + K_{n=1}^{\infty}(d_n z^{\alpha_n}/1)$ ($d_n \neq 0$, α_n a positive integer, $n = 1, 2, 3, \cdots$) represents for $|z| < 1$ a function $f(z)$ which is analytic at $z = 0$ and has only polar singularities in $|z| < 1$ provided $\lim_{n\to\infty} |d_n|^{1/\alpha_n} = 1$, $\lim_{n\to\infty} \alpha_n = \infty$. It is shown that if, in addition, $\limsup_{n\to\infty} h_n/h_{n-1} = \infty$, where $h_n = \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}$, then the function $f(z)$ has the circle $|z| = 1$ as a natural boundary. The function $z = \varphi(w) = w(1 - \alpha w)/(1 - \beta w)$, $0 < \alpha < \beta < 1$, and its inverse, $w = \psi(z)$, $\psi(0) = 0$, are used to show that $F(w) = f[\varphi(w)]$ has a non-polar singularity in $|w| \leqq 1$, from which fact the theorem follows. *W. T. Scott* (Evanston, Ill.).

Wilson, R. On the Hadamard product of two isolated essential points of finite exponential order. J. London Math. Soc. 28, 490–494 (1953).

Let $\varphi_i(z)$ have an isolated essential singularity at $z = 1$ and no other singularities. The same is then true for the Hadamard product $\psi(z)$ of $\varphi_1(z)$ and $\varphi_2(z)$. The author defines order, type and directions of strongest growth (DSG) associated with the essential singularities of $\varphi_i(z)$, $\psi(z)$ to be the corresponding entities for the entire functions $f_i(z) = \varphi_i(1 - z^{-1})$, $h(z) = \psi(1 - z^{-1})$. He expresses order, type and DSG of $\psi(z)$ in terms of those of the $\varphi_i(z)$. The results follow from previous work by the author [same J. 28, 185–193 (1953); these Rev. 14, 739] and from joint work by A. J. Macintyre and the author [ibid. 22, 298–304 (1948); these Rev. 10, 25]. *J. Korevaar* (Madison, Wis.).

Shah, S. M., and Singh, S. K. Borel's theorem on a-points and exceptional values of entire and meromorphic functions. Math. Z. 59, 88–93 (1953).

Let $f(z)$ be an entire function of finite order ρ. The number α is a Borel exceptional value if

$$\limsup \frac{\log^+ n(r, \alpha)}{\log r} < \limsup \frac{\log \log M(r)}{\log r},$$

and if this occurs, ρ is an integer and there is no other exceptional value. The authors first ask whether the same is true with lim sup replaced by lim inf. The answer was known to be "no", but no explicit example had been given; the authors construct one for any irrational ρ, with an arbitrary finite number of "lim inf" exceptional values. Let E-exceptional values be defined as in Shah's work [Compositio Math. 9, 227–238 (1951); Duke Math. J. 19, 585–593 (1952); these Rev. 13, 452; 14, 365]. If $F(z)$ is a meromorphic function of finite order and $T(r, F)$ is its Nevanlinna characteristic function, then $T(r, F') \sim T(r, F)$ if some finite value and ∞ have Nevanlinna defect 1 for F; if two finite values have defect 1, then $T(r, F') \sim 2T(r, F)$. If F has a finite value and ∞ as E-exceptional values, $F^{(k)}$ has 0 and ∞ as E-exceptional values; if F has two finite E-exceptional values, F' has 0 as its only E-exceptional value. These theorems imply corresponding results on Borel exceptional values. *R. P. Boas, Jr.* (Evanston, Ill.).

Boas, R. P., Jr. A Tauberian theorem for integral functions. Proc. Cambridge Philos. Soc. **49**, 728–730 (1953).

Let $f(z)$ be a canonical product of genus 1 with zeros z_n; $n(r)$ gives the number of z_n in $|z| < r$. When the z_n are real the following three conditions are known to be equivalent [Pfluger, Comment. Math. Helv. 16, 1–18 (1944); these Rev. 5, 258]. (A) $\lim y^{-1} \log |f(iy)|$ exists for $y \to \infty$; (D) $\lim r^{-1} n(r)$ exists for $r \to \infty$; (I) $\lim \int_R^R x^{-2} \log |f(x)|\, dx$ exists for $R \to \infty$. What can be said if the z_n instead of being real only satisfy the condition (C) $\sum |\operatorname{Im}(1/z_n)| < \infty$? Pfluger [loc. cit.] showed that (D) and (I) are equivalent. The present author remarks that M. E. Noble has proved in his (unpublished) thesis that (A) and (C) together imply (D). The author himself shows that under the condition (C), condition (D) is equivalent to (A′): $\lim y^{-1} \log |f(iy)|$ exists for $y \to \infty$ if y is excluded from a set of finite logarithmic length. *J. Korevaar* (Madison, Wis.).

Lakshminarasimhan, T. V. A Tauberian theorem for the type of an entire function. J. Indian Math. Soc. (N.S.) **17**, 55–58 (1953).

Let $n(r)$ denote the number of zeros in $|z| \leq r$ of an entire function of order $\rho > 0$ and type τ. Let L and l denote, respectively, the upper and lower limit of $n(r)/r^\rho$ $(r \to \infty)$. The author gives a proof of S. M. Shah's inequality $Le^{1/L} \leq e\rho\tau$ [Proc. Indian Acad. Sci., Sect. A. 28, 519–526 (1948); these Rev. 10, 693]. His method is a refinement of that used by R. C. Buck [J. Indian Math. Soc. (N.S.) 16, 147–149 (1953); these Rev. 14, 631] who obtained a somewhat weaker result. *J. Korevaar* (Madison, Wis.).

Popov, I. V. Remark on the note, "On a question of Prof. V. N. Deputatov". Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 151 (1953). (Russian)

See same Uspehi (N.S.) 6, no. 4(44), 170–171 (1951); these Rev. 13, 334.

Andersson, Bengt J. A note on the constant of Koebe. Ark. Mat. **2**, 415–416 (1953).

Using a lemma of Beurling [Thèse, Uppsala, 1933, p. 44], the author shows that if $w(z) = a_1 z + a_2 z^2 + \cdots$ is schlicht in $|z| < 1$ and maps $|z| < 1$ onto a region D_w of the w-plane, and if $d_w = \inf_{w \text{ non-}\in D_w} |a_1|^{-1} |w|$, $M_w = \sup_{w \in D_w} |a_1|^{-1} |w|$, and $M_w \leq M < \infty$, then $d_w \geq 2M^2[1 - (2M)^{-1} - (1 - M^{-1})^{\frac{1}{2}}]$. Unfortunately, the result is not new; the author seems unaware of the results of G. Pick [Akad. Wiss. Wien, S.-B. IIa. 126, 247–263 (1917)] and R. Nevanlinna [Översikt Finska Vetenskaps-Soc. Förhandlingar **63A**, no. 6 (1921)] and of a recent short proof by A. W. Goodman [Bull. Amer. Math. Soc. 55, 363–369 (1949); these Rev. 10, 601]. *A. J. Lohwater* (Ann Arbor, Mich.).

Iliev, Lyubomir. On triply symmetric univalent functions. Bŭlgar. Akad. Nauk. Izvestiya Mat. Inst. **1**, 27–34 (1953). (Bulgarian)

A translation into Bulgarian, with the addition of more details, of a previously published note [Doklady Akad. Nauk SSSR (N.S.) **84**, 9–12 (1952); these Rev. **13**, 832, 1140]. *A. W. Goodman* (Lexington, Ky.).

Jenkins, James A. Various remarks on univalent functions. Proc. Amer. Math. Soc. **4**, 595–599 (1953).

Let S denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular and univalent for $|z| < 1$, and $F = F(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n)$ be a real-valued function subject to certain conditions of smoothness. Schaeffer and Spencer have shown that every function $f(z) \in S$ belonging to a point $(a_2, \cdots, a_n)$ where F attains its maximum in the nth coefficient region V_n for S satisfies a differential equation of the form

$$(zf'(z)/f(z))^2 \sum_{\nu=1}^{n-1} A_\nu f(z)^{-\nu} = \sum_{\nu=n+1}^{n-1} B_\nu z^{-\nu},$$

A_ν and B_ν being the definite constants depending on $f(z)$ as well as F. The author considers a polynomial $P(a_2, \cdots, a_n)$ satisfying a homogeneity

$$P(a_2 e^{i\theta}, \cdots, a_n e^{i(n-1)\theta}) = e^{ik\theta} P(a_2, \cdots, a_n)$$

with $k \neq 0$, of which the maximum modulus is desired to be found in V_n. He then shows that the only such homogeneous polynomials for which a single term occurs on the left-hand side of the equation are $a_2, a_2^2 - a_3$ and their powers, possibly multiplied by a constant. Analogous results are derived for the classes of functions $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$ univalent for $|z| > 1$ and of functions $\phi(w)$ inverse to the preceding.

Y. Komatu (Tokyo).

Lohwater, A. J., and Piranian, George. On the derivative of a univalent function. Proc. Amer. Math. Soc. **4**, 591–594 (1953).

Let S denote the class of functions $f(z)$ regular and univalent in $|z| < 1$. As shown by Seidel and Walsh [Trans. Amer. Math. Soc. 52, 128–216 (1942); these Rev. 4, 215], every $f(z) \in S$ satisfies $f'(re^{i\theta}) = o((1-r)^{-1/2})$ as $r \to 1$ for all $e^{i\theta}$ on $|z| = 1$ except for a possible set of linear measure zero. The authors prove that there exists a function $f(z) \in S$ and a set E on $|z| = 1$ of positive logarithmic capacity such that $(1-r)^{1/2} |f'(re^{i\theta})| \to \infty$ as $r \to 1$ whenever $e^{i\theta} \in E$.

Y. Komatu (Tokyo).

Suetin, P. K. The theorems of Abel and Tauber for series of Faber polynomials. Doklady Akad. Nauk SSSR (N.S.) **91**, 27–30 (1953). (Russian)

Let $\{a_n\}$ be a sequence of complex numbers, and let K be a continuum whose complement is a simply-connected domain G_∞ containing the point $z = \infty$. Let $z = \Psi(w)$, $\Psi(\infty) = \infty$, $\Psi'(\infty) > 0$, be the univalent function which maps $|w| > 1$ onto G_∞, and let $\Phi_n(z)$ be the Faber polynomials associated with the boundary Γ of K. The author forms the series (*) $f(z) = \sum_0^\infty a_n \Phi_n(z)$ and imposes conditions on Γ so that (*) has many of the properties of power series in the interior region G determined by Γ, including analogues of the Abel and Tauber theorems, of which the following result is typical: If Γ is analytic at some boundary

point z_0, and if (*) converges at z_0, then the sum (*) tends to $f(z_0)=\sum_0^\infty a_n\Phi_n(z_0)$ as $z\to z_0$ along an arbitrary but non-tangential path inside G if one of the following conditions is satisfied: (1) Γ is such that $\Psi''(w)\in H_1$; (2) $\Psi'(w)\in H_1$ and $a_n=O(n^{-1})$; (3) $\Psi'(w)\in H_\delta$ and $\sum_0^\infty|a_n|^\delta$ converges, where $1<\delta\leq 2$; (4) $a_n=O(n^{-1}\epsilon_n)$, where $\epsilon_n>0$ and $\sum_1^\infty \epsilon_n^{2n-1}$ converges. *A. J. Lohwater* (Ann Arbor, Mich.).

Rådström, Hans. On the iteration of analytic functions. Math. Scand. 1, 85–92 (1953).

This work extends to meromorphic functions some of the iteration theory that Fatou [Bull. Soc. Math. France 47, 161–271 (1919); 48, 33–94, 208–314 (1920); Acta Math. 47, 337–370 (1926)] and Julia [J. Math. Pures Appl. (8) 1, 47–245 (1918)] gave for entire functions and rational functions. Let $f(x)$ be meromorphic in a region D of the complex number sphere S. A point $x\in D$ is of infinite order if all the iterates $f^n(x)$ $(n=0, 1, 2, \cdots)$ lie in D; and the set of such points is denoted by D_ω. For convenience it is assumed that if $S-D$ contains an isolated point then this point is an essential singularity of f. (This is not restrictive, since otherwise the isolated point can be adjoined to D.) If $S-D_\omega$ contains more than two points then it contains no isolated point. In consequence there are only four cases: $S-D_\omega$ either I) contains no point; or II) one point; or III) two points; or IV) is dense in itself. When I holds, $D_\omega=S$ and f is rational. Under II, $D_\omega=S-\{a\}$, and $D=S-\{a\}$, and by a suitable linear fractional transformation (hereafter l.f.t.) point a can be carried to ∞, in which case (the transformed) f is an entire function which is not a polynomial. Cases I and II were studied by Fatou and Julia [loc. cit.]; for these cases there always exist points in D_ω at which $\{f^n\}$ is not a normal family.

In Case III, $D_\omega=S-\{a, b\}$, and either III_a) $S-D$ contains just one point, or III_b) just two points. For III_a) $S-D=\{a\}$, we can throw the essential singularity a to ∞ and b to 0 by a l.f.t., and then f has the form $f(x)=x^{-n}e^{F(x)}$ where n is a positive integer and F is a non-constant entire function. For III_b, a and b are both essential singularities of f, and by a suitable l.f.t. (of a to ∞ and b to 0), f becomes $f(x)=x^n e^{F(x)+G(1/x)}$ with n an integer and F, G non-constant entire functions. In Case III, set D_ω contains infinitely many points at which the family $\{f^n\}$ is not normal; whereas in Case IV, $\{f^n\}$ is normal at every interior point of D_ω.

I. M. Sheffer (State College, Pa.).

Hersch, Joseph. Sur une forme générale du théorème de Phragmén-Lindelöf. C. R. Acad. Sci. Paris 237, 641–643 (1953).

The author continues the work of his preceding note on extremal length [same C. R. 235, 569–571 (1952); these Rev. 14, 262]. Asymptotic formulas for modules and harmonic measures are given. Applications are made to Phragmén-Lindelöf theorems. Generalizations to three-dimensional space are indicated. *M. Heins.*

Nehari, Zeev. Some inequalities in the theory of functions. Trans. Amer. Math. Soc. 75, 256–286 (1953).

Starting with a function S harmonic except for assigned singularities in a domain D the author obtains a functional of S which increases with expanding D. A systematic development of this result in extended forms leads the author to relatively simple proofs of a large number of inequalities, some new and some previously obtained by the use of deeper methods. Thus let $z=g(w)=w+b_1/w+b_2/w^2+\cdots$ map a schlicht domain D containing $w=\infty$ onto $|z|>1$, and let $w=z-b_1/z-b_2/z^2-(b_3+b_1^3)/z^3-\cdots$ be the inverse function. Then $|b_1|\leq 1$ is trivial, but the author's method also leads to the sharp $|b_2|\leq 2/3$ due to Schiffer [Bull. Soc. Math. France 66, 48–55 (1938)] and Golusin [Mat. Sbornik N.S. 3(45), 321–330 (1938)] and $|b_3|\leq 1$ due to Springer [Trans. Amer. Math. Soc. 70, 421–450 (1951); these Rev. 13, 24]. Also if, for fixed D, $g(w)$ is an arbitrary univalent function in D the domain of variability of $2b_3+b_1^3$ is a circle.

Among other new results we may quote the following. Let $f(z)$, $g(z)$ give schlicht maps of the unit circle onto two mutually disjoint domains. Then

$$|f'(0)|\cdot|g'(0)|\leq|f(0)-g(0)|^2$$

with equality only if the domains are complementary half-planes, whose dividing line is the perpendicular bisector of the line joining $f(0)$, $g(0)$. The corresponding result for bounded schlicht functions is also proved.

Finally let D be a simply connected domain, a a fixed point of D and $R(a)$ the inner radius of D at a. If $2m$ replicas of D are placed in the plane without overlapping and $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_m$ denote the corresponding positions of a, then

$$R(a)\leq\left\{\frac{\prod_{\mu,\nu=1}^{m}|\eta_\mu-\xi_\nu|}{\prod_{\mu<\nu}|\xi_\mu-\xi_\nu|\prod_{\mu<\nu}|\eta_\mu-\eta_\nu|}\right\}^{\frac{1}{m}}.$$

When D is a square, triangle or parallel strip, and a is centrally placed, the inequality is asymptotically best possible as $m\to\infty$. *W. K. Hayman* (Exeter).

Lehto, Olli. A majorant principle in the theory of functions. Math. Scand. 1, 5–17 (1953).

The author's results are either special cases or simple consequences of Theorems 210, 214 of J. E. Littlewood's "Lectures on the theory of functions" [Oxford, 1944; these Rev. 6, 261] and the Ahlfors-Shimizu identity.

W. K. Hayman (Exeter).

***Jenkins, James A. Some results related to extremal length.** Contributions to the theory of Riemann surfaces, pp. 87–94. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The author illustrates the method of extremal lengths by using it to solve a variety of simple problems in conformal mapping. In some cases his results are more general than those previously obtained by other authors. In particular, the range of the method is not limited to univalent functions.

W. K. Hayman (Exeter).

Jenkins, James A. Symmetrization results for some conformal invariants. Amer. J. Math. 75, 510–522 (1953).

The author solves the following problem. Given the class S of $f(z)=z+b_2z^2+\cdots$ regular and schlicht in $|z|<1$, and $|f(-r_1)|$, to maximize $|f(r_2)|$, where r_1, r_2 are fixed and $0<r_1, r_2<1$. An extremal function is obtained explicitly in terms of Schwarz-Christoffel mappings and turns out to have real coefficients. It follows that if $r_1\leq r_2$

$$|f(-r_1)|+|f(r_2)|\leq\frac{r_1}{(1+r_1)^2}+\frac{r_2}{(1-r_2)^2},$$

generalizing the corresponding result proved by Golusin when $r_1=r_2$ [Mat. Sbornik N.S. 18(60), 379–390 (1946); these Rev. 8, 574] which itself yields Löwner's inequality $|b_3|\leq 3$. [Math. Ann. 89, 103–121 (1923)]. The method of

proof has particular interest in that it does not use Löwner's differential equation, but the notion of modules for pentagons, previously developed by the author [Trans. Amer. Math. Soc. 67, 327–350 (1949); these Rev. 11, 341] and the fact that these behave monotonically under circular symmetrisation. *W. K. Hayman* (Exeter).

Meschkowski, Herbert. Verzerrungssätze für mehrfach zusammenhängende Bereiche. Compositio Math. 11, 44–59 (1953).

Let F be the class of functions which are regular and single valued in a schlicht, bounded region $\mathfrak{B}$ and on its boundary $\mathfrak{C}$ consisting of n smooth curves. Let $\varphi_\nu(z)$, $\nu=1, 2, \cdots$ be a complete system of orthonormal functions for the class F' consisting of derivatives of functions in F, with orthonormality with respect to the inner product

$$(f', g') = (1/2i)\int_{\mathfrak{C}} f'\bar{g}dz.$$

Assume that $f(z)$ gives a schlicht map of $\mathfrak{B}$ which carries $v \in \mathfrak{B}$ into $w=\infty$ with residue 1 at $z=v$. For an arbitrary point $u \in \mathfrak{B}$, let

$$c_\nu(u) = ((z-u)^{-1}, \varphi_\nu(z)), \quad D_\nu(z) = \int_v^z c_\nu(t)dt,$$

$$\psi_\nu(z) = \int_v^z \varphi_\nu(t)dt.$$

The author derives the following distortion theorem:

$$\frac{\exp\{-R\sum D_\nu(u)\psi_\nu(u) - \pi\sum|\psi_\nu(u)|^2\}}{|u-v|^2} \leqq |f'(u)| \leqq \frac{\exp\{-R\sum D_\nu(u)\psi_\nu(u) + \pi\sum|\psi_\nu(u)|^2\}}{|u-v|^2}.$$

He shows that this leads to a known theorem [Bieberbach, Einführung in die konforme Abbildung, 4. Aufl., Springer, Berlin, 1949, p. 119] for the exterior of the unit circle. He then evaluates the quantities appearing in the theorem for the case of an annulus. *G. Springer* (Evanston, Ill.).

***Lewy, Hans. Asymptotic developments at the confluence of boundary conditions.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 255–256. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

This is a brief statement of results previously published [Univ. California Publ. Math. (N.S.) 1, 247–280 (1950); these Rev. 12, 691]. *W. Kaplan* (Ann Arbor, Mich.).

af Hällström, Gunnar. Eine quasikonforme Abbildung mit Anwendungen auf die Wertverteilungslehre. Acta Acad. Aboensis 18, no. 8, 16 pp. (1952).

In this paper the author considers various quasi-conformal mappings to obtain results on the variation in Robin constants, modulus of a ring, Nevanlinna characteristics, etc.

Let G denote a ring-shaped domain, with inner contour Γ_0, outer contour Γ_1, in the z-plane, with modulus M. If G is mapped conformally onto a rectangle $2\pi\times M$ in the w-plane, then $2\pi M = \int_G d\sigma_w$, where $d\sigma_w$ is the element of area in the w-plane. Now let G', a domain the z'-plane, be the quasi-conformal image of G. If $Q(z)$ is the dilation quotient of the map and if M', $d\sigma_{w'}$ are defined as above (for G'), then

$$\frac{M}{1+(I/M)} \leqq M' \leqq M+I,$$

where $I=(1/2\pi)\int\int_G(Q(z)-1)d\sigma_w$. It follows that M differs from M' by at most I. Here I is a sharper double integral analogue of a result due to Pfluger [Ann. Inst. Fourier Grenoble 2, 69–80 (1951); these Rev. 13, 453].

Let H be a domain bounded by a continuum Γ, and containing $z=\infty$, and let H' be a quasi-conformal image of H such that $z=\infty$ corresponds to $z'=\infty$, such that Γ' is the image of Γ, and such that $J=(1/2\pi)\int\int_H(Q(z)-1)d\sigma_w$ remains finite. Now if the mapping of H on H' is a normalized almost conformal mapping in the sense of Thiem [Comment. Math. Helv. 23, 26–49 (1949); these Rev. 11, 22], then the Robin constants of Γ and Γ' differ by at most J. If H is the domain H_r: $[z\,|\,|z|>r]$, and if

$$J(r) = \frac{1}{2\pi}\int\int_{H_r}(Q(z)-1)\frac{d\sigma_z}{r^2},$$

then the normalized almost-conformal image of the circle $|z|=r$ lies in the ring $re^{-J(r)} \leqq |z'| \leqq re^{J(r)}$.

The author also generalizes slightly a quasi-conformal mapping introduced by Kakutani [Jap. J. Math. 13, 375–392 (1937)] and makes several applications of this mapping. He applies it to two examples in the distribution of values of a meromorphic function of finite order, and of infinite order; here the problems considered are analogous to those considered by Thiem and af Hällström [af Hällström, Acta Acad. Aboensis 12, no. 8 (1940); these Rev. 2, 275].

M. O. Reade (Ann Arbor, Mich.).

Künzi, Hans P. Über ein Teichmüllersches Wertverteilungsproblem. Arch. Math. 4, 210–215 (1953).

This is a detailed investigation of a Riemann surface whose Speiser graph consists of one half of the graph of the Weierstrass γ-function. The main tool is a lemma of Teichmüller and Wittich [see Wittich, Math. Z. 51, 278–288 (1949); these Rev. 10, 241]. *L. V. Ahlfors.*

Fuks, B. A. Natural boundaries of analytic functions of complex variables. Amer. Math. Soc. Translation no. 93, 59 pp. (1953).

Translated from Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 75–120 (1950); these Rev. 12, 252.

Tornehave, Hans. On analytic functions of several variables. A theorem on analytic continuation. Math. Scand. 1, 73–81 (1953).

Let ω be a point-set in the quadrant $x\geqq 0$, $y\geqq 0$ of the (x, y)-plane. The author defines the hyperbolic completion of ω to be the point-set ω^* consisting of all points (x, y) which satisfy the conditions $x^2-y^2=x_0^2-y_0^2$, $0\leqq x\leqq x_0$, $0\leqq y\leqq y_0$ for a point (x_0, y_0) of ω. As his main theorem, the author proves the following result. Every function $f(z_1, \cdots, z_m)$ $(m\geqq 2)$ of m complex variables $z_j=x_j+iy_j$ which is analytic in a domain containing the point set $(\|\mathbf{x}\|, \|\mathbf{y}\|) \in \omega$ possesses an analytic continuation into a domain which contains the point-set $(\|\mathbf{x}\|, \|\mathbf{y}\|) \in \omega^*$. (Here $\|\mathbf{x}\| = (x_1^2+\cdots+x_m^2)^{1/2}$ and $\|\mathbf{y}\| = (y_1^2+\cdots+y_m^2)^{1/2}$.)

The author bases the proof upon two auxiliary theorems which he proves, the first of which is one of the Hartogs' type for functions of $m+1$ complex variables $z_1, \cdots, z_m, w$ and the second of which is a result on analytic completion

in the space of $m+2$ complex variables $z_1, \cdots, z_m, w_1, w_2$. The first of these auxiliary theorems involves the use of Laurent series in w rather than Taylor series as in Hartogs' classical result [Math. Ann. 62, 1–88 (1906)].

W. T. Martin (Cambridge, Mass.).

Ozaki, Shigeo, and Onô, Isao. Analytic functions of several complex variables. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 262–270 (1953).

By repeated use of results of analytic functions of one complex variable, the authors give versions of Cauchy's integral formula, Taylor's and Laurent's expansions, and the maximum principle for systems of analytic functions of n complex variables. The results are stated very concisely in vector and matrix notation. *W. T. Martin.*

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. Note on vector spaces. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 271–276 (1953).

The authors give simple sufficient conditions for the univalency in $r \leqq |Z| \leqq R$ of a function $W(Z)=G(Z)+H(Z)$, where G and H are regular analytic functions, G in $|Z| \geqq r$ and H in $|Z| \leqq R$. Here

$$Z=\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

and W are n-column vectors, and $|Z|$ is the norm $[\sum |z_j|^2]^{1/2}$.

W. T. Martin (Cambridge, Mass.).

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. Note on normed rings. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 277–282 (1953).

The authors first define (left) analyticity in a (non-commutative) normed ring R. They then obtain a sufficient condition for the univalency of such an analytic function in a convex domain K of R. If an n-column vector

$$Z=\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

is written as a square matrix

$$Z=\begin{pmatrix} z_1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ z_n & 0 & \cdots & 0 \end{pmatrix}$$

and if the customary norm for matrices is used, then the authors point out that in the sense of this norm a vector space becomes a (non-commutative) normed ring R. Having this, they obtain as corollaries to their main theorem two results giving sufficient conditions for the univalency of an analytic function of an n-column vector Z. The first of these corollaries is the same as one obtained earlier by the same authors [same Rep. 4, 230–237 (1952); these Rev. 14, 368] and the second one represents a stronger result than the one originally given by S. Takahashi [Ann. of Math. (2) 53, 464–471 (1951), theorem 1; these Rev. 12, 818].

W. T. Martin (Cambridge, Mass.).

Fréchet, Maurice. Les fonctions "para-analytiques" à n dimensions. C. R. Acad. Sci. Paris 236, 1832–1834 (1953).

The author begins by defining a real n-dimensional commutative algebra with identity which he calls a para-complex number system. He then defines a concept of differentiation of functions on E^n with values in the algebra which yields a class of functions called para-analytic. This concept reduces to classical analytic functions when the algebra is the complex numbers. Some higher dimensional examples are given. *C. J. Titus* (Ann Arbor, Mich.).

Caccioppoli, Renato. Fondamenti per una teoria generale delle funzioni pseudo-analitiche di una variabile complessa. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 13, 197–204, 321–329 (1952).

The author gives concise proofs of theorems sketched in C. R. Acad. Sci. Paris 235, 228–229 (1952) [these Rev. 14, 364]. In the review referred to there is a misprint for condition (i) which should read $\Phi \geqq \mu\overline{\Phi}$ and a misprint in the original paper, also part of (i) which should read $J \geqq 0$.

C. J. Titus (Ann Arbor, Mich.).

Theory of Series

Sage de Romaña, M. On a generalization of the quotient criterion of Weierstrass concerning convergence and divergence of infinite series. Univ. Nac. Tucumán. Revista A. 9, 37–44 (1952). (Spanish)

It is shown that if $\sum a_n$ and $\sum b_n$ are series of complex numbers for which $b_{n+1}/b_n \to 1$ and $a_{n+1}/a_n = (b_{n+1}/b_n)+\gamma_n$ where $\sum |\gamma_n| < \infty$, then $\sum a_n$ and $\sum b_n$ are both convergent or both divergent. While the author does not note the fact, his hypotheses imply (but are not implied by) the hypothesis that $a_{n+1}/a_n = (b_{n+1}/b_n)(1+c_n)$ where $\sum |c_n| < \infty$. Hence his result is implied by a theorem of H. Jehle [Math. Z. 52, 60–61 (1949); these Rev. 11, 241] for which it was shown by Agnew [Pacific J. Math. 1, 1–3 (1951); these Rev. 13, 226] that the hypothesis $\sum |c_n| < \infty$ cannot be weakened. The theorem of the author is strong enough to furnish proofs of theorems of Gauss and Weierstrass on convergence of series, and elegant generalizations of these theorems. *R. P. Agnew* (Ithaca, N. Y.).

Besicovitch, A. S. On rearrangement of conditionally convergent series. J. London Math. Soc. 28, 480–483 (1953).

Assuming the hypothesis of the continuum, the author proves the following proposition. There exists a series $\sum_{n=0}^{\infty} u_n(x)$ such that (a) for each real x, the series is either a conditionally convergent series of real terms or a rearrangement of such a series and (b) each rearrangement converges to zero for all values of x except for values of x in a set which is at most enumerable. The proof begins with an arrangement of all nonnegative monotone integral-valued functions $f(n)$ tending to $+\infty$ into a transfinite sequence $f_i(n)$, $i<\omega_1$, and depends upon use of transfinite numbers. A simple modification of the proof establishes another proposition obtained by replacing the words "converges to zero" of (b) by the word "diverges". *R. P. Agnew.*

Nasr, Saad Khalil. On sequences associated with a given series of positive terms and functions associated with a given integral of a positive continuous function. Proc. Math. Phys. Soc. Egypt 4 (1952), no. 4, 143–153 (1953).

Let $\sum p_n$ be a series of positive terms with partial sums $P_n = p_1+p_2+\cdots+p_n$ and remainders $R_n = p_{n+1}+p_{n+2}+\cdots$. After giving classic theorems concerning $\sum p_n/P_n^{\alpha}$ and $\sum p_n/R_n^{\alpha}$, the author gives the following theorem and related ones. Let $\sum p_n = \infty$, let k be a positive integer, and let $p_n/P_{n-k} \to B^{(k)}$ as $n \to \infty$. Then $p_n/P_n \to x$ as $n \to \infty$ where

x is a solution of the equation $x/(1-x)^k = B^{(k)}$. Analogous theorems, involving integrals of positive functions, are given. *R. P. Agnew* (Ithaca, N. Y.).

Kuttner, B. A theorem on Riesz summability. J. London Math. Soc. **28**, 451–461 (1953).

The following theorem is proved. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of nonnegative numbers increasing to infinity, and let $\nu_n = c\lambda_n + d\mu_n$ where c and d are positive constants. Let $0<k<1$ and $\delta>0$ or let $k=1$ and $\delta \geqq 0$. If a series $\sum a_n$ is evaluable (R, λ_n, k) to L and also evaluable (R, μ_n, k) to L, then it is likewise evaluable $(R, \nu_n, k+\delta)$ to L. Examples are given to show that the conclusion fails if $k>1$ and if $0<k<1$ while $\delta=0$. *R. P. Agnew* (Ithaca, N. Y.).

Ramanujan, M. S. Series-to-series quasi-Hausdorff transformations. J. Indian Math. Soc. (N.S.) **17**, 47–53 (1953).

Let $\{\mu_n\}$ be a sequence of numbers; let $A=(a_{nk})$ be the Hausdorff matrix generated by $\{\mu_n\}$; and let B be the transpose of A. The author finds a condition on $\{\mu_n\}$ which is necessary and sufficient in order that the series-to-series transformation represented by B should sum every convergent series to its ordinary sum. Expressed directly in terms of $\{\mu_n\}$, the condition is this, that μ_n is a moment constant and that $\mu_0=1$. In terms of A, the condition is this, that $\sup \sum_k |a_{nk}| < \infty$ and $\sum_k a_{nk} = 1$. *G. Piranian.*

Tropper, A. Mary. A sufficient condition for a regular matrix to sum a bounded divergent sequence. Proc. Amer. Math. Soc. **4**, 671–677 (1953).

The author proves the following result. In order that the regular normal matrix A shall sum a bounded divergent sequence it is sufficient that (a) its unique reciprocal B shall not be regular, and (b) there exists a normal matrix Q with $\|Q\| < \infty$ (if $Q=(q_{nk})$ then $\|Q\| = \sup_n \sum_k |q_{nk}|$), whose columns are all null sequences such that $C=BQ$ has bounded columns nd $\|C\| = \infty$. The proof consists in constructing a bounded divergent sequence whose A transform is a null sequence. If we take $Q=I$, the identity matrix, condition (b) may be replaced by the assumption that all the columns of B form bounded sequences. These results are closely related to those obtained by Agnew [Bull. Amer. Math. Soc. **52**, 128–132 (1946); these Rev. **7**, 292] and Wilansky [ibid. **55**, 914–916 (1949); these Rev. **11**, 243]. A new and simple proof is given of the well known theorem due to Brudno [Mat. Sbornik N.S. **16(58)**, 191–247 (1945); these Rev. **7**, 12] to the effect that if A is a regular matrix there exists a normal matrix A^*, such that A and A^* are mutually consistent for bounded sequences. *V. F. Cowling.*

Sonnenschein, J. Sur une classe de procédés de sommation. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 537–542 (1953).

For some special kernels $H(t, \alpha)$, it is shown that if $\varphi(t)$ is real, bounded, and measurable over each finite interval $0 \leqq t \leqq t_0$ and if $\varphi(t) \to L$ as $t \to \infty$, then

$$\sigma(\alpha) = \int_0^\infty H(t, \alpha)\varphi(t)dt$$

exists for each $\alpha>0$ and $\sigma(\alpha) \to L$ as $\alpha \to \infty$. The proofs could be shortened by use of the theory of kernel transformations expounded, for example, by Hardy [Divergent series, Oxford, 1949; these Rev. **11**, 25]. *R. P. Agnew.*

Rajagopal, C. T. On Tauberian oscillation theorems. Compositio Math. **11**, 71–82 (1953).

For kernels $\psi(x, t)$ belonging to a specified class and for functions $s(t)$ satisfying a stated unilateral Tauberian condition of Schmidt type, it is shown that the transform

$$\Psi(x) = \int_0^\infty \psi(x, t)s(t)dt$$

is such that

$$\liminf_{x\to\infty} \Psi(x) = \liminf_{t\to\infty} s(t)$$

and

$$\limsup_{x\to\infty} \Psi(x) = \limsup_{t\to\infty} s(t).$$

The result is applied to Borel transforms. *R. P. Agnew* (Ithaca, N. Y.).

Tatchell, J. B. On some integral transformations. Proc. London Math. Soc. (3) **3**, 257–266 (1953).

With a given kernel $\varphi(s, t)$, the transform

$$\alpha_s(x) = \int_0^\infty \varphi(x, t)ds(t)$$

of a function $s(t)$ is said to exist if the integral in

$$\alpha_{s,h}(x) = \int_0^h \varphi(x, t)ds(t)$$

exists for each $h>0$ as a Riemann-Stieltjes integral and $\alpha_{s,h}(x) \to \alpha_s(x)$ as $h \to \infty$. One theorem gives three conditions on $\varphi(x, t)$ which are necessary and sufficient to ensure that $\alpha_s(x)$ exists for each $x \geqq 0$ and is Lebesgue integrable over $x \geqq 0$ whenever $s(t)$ has bounded variation over $t \geqq 0$. Another theorem gives two conditions on $\varphi(x, t)$ which are necessary and sufficient to ensure that $\alpha_s(x)$ exists for each $x \geqq 0$ and has bounded variation over $x \geqq 0$ whenever $s(t)$ has bounded variation over $t \geqq 0$. A third theorem characterizes kernels $K(x, t)$ for which $\beta_v(x) = \int_0^\infty K(x, t)v(t)dt$ exists as a Lebesgue (or Cauchy-Lebesgue) integral for each $x \geqq 0$ and is Lebesgue integrable over $x \geqq 0$ whenever $v(t)$ is Lebesgue integrable over $x \geqq 0$. As the author points out in an addendum, his conditions are complicated by the fact that $K(x, t)$ is not assumed to be (and in fact need not be) a measurable function of its two arguments. *R. P. Agnew* (Ithaca, N. Y.).

Tanaka, Chuji. On the convergence-abscissas of the generalized factorial series. Proc. Amer. Math. Soc. **4**, 150–160 (1953).

Series of the complex variable $s=\sigma+it$ of the form

$$F(s) = \sum_{n=1}^{\infty} a_n[\lambda_1\lambda_2\cdots\lambda_n][(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)]^{-1}$$

are studied, where

$$\lambda_n = r_n e^{i\phi_n}, \quad \lim_{n\to\infty} r_n = +\infty, \quad |\phi_n| \leqq \phi < \pi/2 \quad (n=1, 2, \cdots).$$

The case of real exponents ($\phi_n=0$) is treated in greater detail. The results for the latter case generalize those of Landau [S.-B. Math.-Phys. Kl. Akad. Wiss. München **36**, 151–218 (1906)], who assumed in addition that $\sum_{n=1}^{\infty} 1/r_n = \infty$, on the existence of abscissas of convergence and absolute convergence of factorial series. Explicit formulas are also given for these convergence abscissas, similar to those which have been derived by Kojima [Tôhoku Math. J. **6**, 134–139 (1914)] and Fujiwara [ibid. **6**, 140–142 (1914)] for generalized Dirichlet series. Fort's

equiconvergence theorem [Trans. Amer. Math. Soc. 31, 233–240 (1929)] is a consequence of these results. The proofs involve analogues of various known theorems concerning ordinary factorial series to generalized factorial series, both with real and complex exponents, satisfying the above conditions. These theorems treat convergence, absolute convergence, uniform convergence, and behavior in sectors and half-planes of the above series. *B. Lepson.*

Fourier Series and Generalizations, Integral Transforms

Mohanty, R. Absolute Cesàro summability of a series associated with a Fourier series. Bull. Calcutta Math. Soc. 44, 152–154 (1952).

Let $f(t)$ have Fourier series $\sum_1^\infty A_n(t)$ and let

$$\varphi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\}.$$

The author shows that if $\int_0^\pi \log kt^{-1}|d\varphi(t)|<\infty$ for some $k>\pi$, then $\sum_1^\infty A_n(t)\log n$ is summable $|C,\delta|$ at $t=x$, for every $\delta>0$. *P. Civin* (Princeton, N. J.).

Hirschman, I. I., Jr. Fractional integration. Amer. J. Math. 75, 531–546 (1953).

Let the function $u(\vartheta)\sim\sum_{-\infty}^{\infty}a_ne^{in\vartheta}$, where $a_0=0$, belong to L^p, $p\geq 1$. Its fractional integral of order $\alpha>0$ is then defined by $u_\alpha(\vartheta)\sim\sum_{-\infty}^{\infty}a_n(in)^{-\alpha}e^{in\vartheta}$. Let also, for all $0\leq\rho<1$ and all α, $u(\rho,\vartheta)=\sum_{-\infty}^{\infty}a_n\rho^{|n|}e^{in\vartheta}$ and $u_\alpha(\rho,\vartheta)=\sum_{-\infty}^{\infty}a_n(in)^{-\alpha}\rho^{|n|}e^{in\vartheta}$. The following results are proved. I. If $1<p<\infty$ and $0<\alpha<1$, then

$$A'\|u\|_p^p\leq\int_0^{2\pi}d\vartheta\left[\int_0^{2\pi}|u_\alpha(\vartheta+\tau)-u_\alpha(\vartheta-\sigma)|^2\tau^{-2\alpha-1}d\tau\right]^{1/2p}\leq A''\|u\|_p^p,$$

where $\|u\|_p=[\int_0^{2\pi}|u|^pd\vartheta]^{1/p}$ and A' and A'' are positive constants depending only on p and α. II. If $1<p<\infty$ and $-\infty<\alpha<1$, then

$$A'\|u\|_p^p\leq\int_0^{2\pi}d\vartheta\left[\int_0^{2\pi}(1-\rho)^{1-2\alpha}|u_{\alpha-1}(\rho,\vartheta)|^2d\rho\right]^{1/2p}\leq A''\|u\|_p^p.$$

The proofs are based on earlier results by Littlewood, Paley, and Zygmund [Littlewood and Paley, Proc. London Math. Soc. (2) 42, 52–89 (1936); Zygmund, Trans. Amer. Math. Soc. 55, 170–204 (1944); these Rev. 5, 230; cf. also, Zygmund, Trigonometrical series, Warsaw-Lvov, 1935, pp. 222–233]. *W. W. Rogosinski* (Newcastle-on-Tyne).

Korevaar, Jacob. Best L_1 approximation and the remainder in Littlewood's theorem. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 281–293 (1953).

Soit $J_0(a,b)$ la classe des fonctions définies et continues pour $a\leq x\leq b$ (sauf en un nombre fini de points où les discontinuités sont de première espèce) et qui satisfont à une condition de Lipschitz d'ordre 1 sur chaque segment partiel, soit $J_m(a,b)$ la classe des fonctions pour lesquelles $f^{(m)}\in J_0(a,b)$. L'auteur précisant des résultats de D. Jackson [The theory of approximation, Amer. Math. Soc. Colloq. Publ., vol. 11, New York, 1930] et de S. Nikolsky [Doklady Akad. Nauk SSSR (N.S.) 58, 185–188 (1947); ces Rev. 9, 282] démontre entre autres les résultats suivants: si $f\in J_0(-\pi,+\pi)$, on peut trouver une somme trigonométrique $T_n(x)$ telle que $\int_{-\pi}^{+\pi}|f(x)-T_n(x)|dx<M/(n+1)$, les coefficients a_{nk}, b_{nk} de T_n étant majorés en module par M'/K (M et M' indépendants de n). Ce résultat est généralisé pour la classe J_m et l'auteur étudie l'approximation en moyenne de $f\in J_m(a,b)$ par des polynomes entiers, puis applique cette méthode à la sommation des intégrales.

M. Zamansky (Paris).

Sz.-Nagy, Béla. Approximation properties of orthogonal expansions. Acta Sci. Math. Szeged 15, 31–37 (1953).

Let $C_{r\alpha}$ ($r=0,1,2,\cdots$; $0<\alpha\leq 1$) be the class of r times differentiable functions on $[0,1]$ for which $f^{(r)}$ satisfies a Lipschitz condition of order α, and set

$$N_{r\alpha}(f)=\sup\frac{|f^{(r)}(x)-f^{(r)}(t)|}{|x-t|}.$$

Extending a method previously used by the reviewer [Duke Math. J. 19, 1–4 (1952); these Rev. 13, 646], the author obtains the following lower bounds for the approximation of f by its Fourier series in the L^p-norm ($1\leq p\leq\infty$): Let $\{\varphi_i\}$ be any orthonormal set on $[0,1]$, let $\{\lambda_{ni}\}$ be a set of numbers such that $|\lambda_{n1}|^2+\cdots+|\lambda_{nn}|^2\leq n$, let $s_n(f)=\sum_{i=1}^n\lambda_{ni}c_i\varphi_i(x)$, where $c_i=(f,\varphi_i)$, and put

$$\rho_n(r,\alpha,p)=\sup\frac{\|f-s_n(f)\|_p}{N_{r\alpha}(f)}\quad(0<N_{r\alpha}(f)<\infty).$$

Then there exist positive constants $\gamma_{r\alpha}$ such that

$$\rho_n(r,\alpha,p)\geq\gamma_{r\alpha}n^{-r-\alpha}.$$

Without further restrictions on λ_{ni}, the result is best possible; this is also true if $\lambda_{ni}\equiv 1$ and $r=0$; the case $\lambda_{ni}\equiv 1$, $r>0$ is left open. If $N_{r\alpha}(f)$ is replaced by the modulus of continuity of $f^{(r)}$, an analogous result is obtained.

W. Rudin (Rochester, N. Y.).

Matsumura, Yoshimi. Note on the summability of orthogonal series. J. Osaka Inst. Sci. Tech. Part I. 3, 21–24 (1951).

Let $\sum_{n=1}^\infty a_n\varphi_n(t)$ be the Fourier expansion of $f(t)\in L^2(a,b)$ relative to the orthonormal system $\{\varphi_n(t)\}$ and let $S_n(t)$ denote the partial sums. Then if $\sum_{n=1}^\infty a_n\varphi_n(t)$ is $(C,1)$ summable almost everywhere so is $\{S_n^k(t)\}$, $k=1,2,\cdots$. For ordinary Fourier series the result is true for $f(t)\in L^1$ [Izumi and Kawata, Tôhoku Math. J. 46, 154–158 (1939); these Rev. 1, 225]. *P. Civin* (Princeton, N. J.).

Hewitt, Edwin. Representation of functions as absolutely convergent Fourier-Stieltjes transforms. Proc. Amer. Math. Soc. 4, 663–670 (1953).

Let G be a locally compact Abelian group and G^* its dual. It is shown that if every bounded, uniformly continuous complex function on G^* is a Fourier-Stieltjes transform, i.e., of the form $\int(y,x)d\varphi(x)$, where φ is a bounded Radon measure on G and (y,x) denotes the value of the character y at x, then G is a finite group. This holds under the weaker condition that every almost periodic function on G^* is a Fourier-Stieltjes transform. *J. L. B. Cooper* (Cardiff).

Obreškov, Nikola. On some integral representations of real functions. Bŭlgar. Akad. Nauk. Izvestiya Mat. Inst. 1, 83–110 (1953). (Bulgarian. Russian summary)

The author proves several inversion and representation theorems for the generalized Laplace transform which he has considered before [Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1., Partie II. 47, 109–134 (1952); these Rev. 14, 1068]. He finds that the kernel is expressible in terms of a Hankel function of imaginary argument [cf.

Magnus and Oberhettinger, Formeln und Sätze für die speziellen Funktionen der mathematischen Physik, 2d ed., Springer, Berlin, 1948, p. 39; these Rev. 10, 38]. [Hence his transform is the so-called Meijer transform; for references see Hirschman and Widder, Trans. Amer. Math. Soc. 66, 135–201 (1949); these Rev. 11, 350.] *R. P. Boas, Jr.*

Korenblyum, B. I. On lacunary Laplace-Stieltjes integrals. Doklady Akad. Nauk SSSR (N.S.) 76, 779–782 (1951). (Russian)

The author proves a Tauberian theorem which generalizes the Hardy-Littlewood "high indices" theorem to Laplace-Stieltjes integrals. Let $S(t)=\int_0^\infty e^{-ts}ds(x)$, $t>0$, where $s(x)$ is continuous on the right, $s(0)=0$, $s(x)$ is of bounded variation on every finite interval, and $s(x)$ is constant in intervals (λ_n, λ_n'), where $\lambda_n'/\lambda_n \geqq \theta > 1$, $\lambda_n' \leqq \lambda_{n+1}$. Then $S(t)\to s(t\to 0^+)$ implies $s(x)\to s(x\to\infty)$ provided that the oscillation of $s(x)$ on $[\lambda_n, \lambda_{n+1}]$ is $|s(\lambda_{n+1})-s(\lambda_n)|$, while the oscillation on $[x, \lambda_{n+1}]$ $(\lambda_n \leqq x < \lambda_{n+1})$ does not exceed

$$\{1-\exp[-\delta(\lambda_{n+1}-x)]\}^{-1}|s(\lambda_{n+1})-s(x)|$$

(for some $\delta>0$). *R. P. Boas, Jr.* (Evanston, Ill.).

on order

***Funk, Paul, Sagan, Hans, und Selig, Franz. Die Laplace-Transformation und ihre Anwendung.** Franz Deuticke, Wien, 1953. vii+106 pp. $2.40.

Beginning with a discussion of elementary functional transformations and the basic operational properties of the Laplace transformation, this book introduces the reader to Heaviside's expansion formula, the operational treatment of systems of ordinary differential equations, the convolution property, the complex inversion formula, illustrative applications to problems in electric circuits and transmission lines and to Abel's integral equation. Asymptotic properties of the transformation are also included. The book concludes with sections on transfer functions and impedance of networks, an introduction to the theory of servomechanisms, and a sketch of the history of the Laplace transformation. Throughout the book most of the mathematical theory is omitted, but adequate references are given for that material. The authors' objective of giving a clear, concise, and useable account of this material, leaving the mathematical details to other books, seems to be realized here with some success. *R. V. Churchill* (Ann Arbor, Mich.).

Bose, B. N. On certain theorems in operational calculus. Bull. Calcutta Math. Soc. 44 (1952), 93–110 (1953).

Let $\phi(p)=pL\{f(t)\}$ where L denotes the Laplace transformation with parameter p. If $\phi(x)$ is self-reciprocal under the Fourier sine transformation on the semiinfinite interval $x>0$, then $f'(x)$ is self-reciprocal under the cosine transformation on that interval and conversely, provided that $f'(x)$ is continuous, that $f(0)=f(\infty)=0$ and that $f(x)$ is absolutely integrable on the interval. This theorem and a number of similar theorems, some involving the functions $\phi(p)$ and $h(t)$, where $f=pL\{g\}$ and $g=pL\{h\}$, and some concerning the self-reciprocal character under Hankel transformations, are derived from known operational properties of Laplace transforms. *R. V. Churchill.*

Chakrabarty, N. K. On certain theorems in operational calculus with two variables. Ganita 4, 1–11 (1953).

This paper contains a number of "rules", and one "sequence", of operational calculus in two variables. Each general result is illustrated by examples. [The author does not seem to know D. Voelker and G. Doetsch's "Die zweidimensionale Laplace-Transformation" [Birkhäuser, Basel, 1950; these Rev. 12, 699], which contains many of his results.] *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Jankowski, W. Sur les zéros des polynomes contenant des parametres arbitraires. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 5 (1951), 31–92 (1953). (Polish and Russian summaries)

Let $P_k(z)$ denote a polynomial of degree p_k having all its zeros in the circle $|z|\leqq r_k$ and let

$$f_k(x)=P_1(x)+a_2P_2(x)+\cdots+a_kP_k(x),$$

where the a_k are arbitrary parameters and $p_1<p_2<\cdots<p_k$. Biernacki [Bull. Internat. Acad. Sci. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1927, 541–685 (1928)] showed $f_2(z)$ to have at least p_1 zeros in the circle $|z|\leqq\max[r_2, (p_2r_1+p_1r_2)/(p_2-p_1)]$. The present author proves similarly that $f_3(z)$ has at least p_1 zeros in

$$|z|\leqq R_3(p_1, p_2, p_3; r_1, r_2, r_3)$$

and that $f_4(z)$ has at least p_1 zeros in

$$|z|\leqq R_4(p_1, p_2, p_3, p_4; r_1, r_2, r_3, r_4).$$

The author specifies these functions R_3 and R_4, but they are rather complicated. In the special case

$$f_3(z)=(z+b)^p+a_2z^{p+1}+a_3z^{p+2},$$

where b and (a_3/a_2) are real, he shows that at least p zeros lie in $|z|\leqq\{2(p+1)+[2p(p+1)]^{\frac{1}{2}}\}|b|/2$, this being an attainable limit. The proofs are similar to Biernacki's, being based upon the principle of argument, as applied in example of $f_3(z)$ to the variation of $\arg[(P_1+a_1P_2)/P_3]$ along a certain circle. *M. Marden* (Milwaukee, Wis.).

Ćetković, Simon. Sur les zéros réels des dérivées d'une classe des fonctions. Bull. Soc. Math. Phys. Serbie 5, no. 1–2, 47–51 (1953). (Serbo-Croatian. French summary)

From the French summary: Let $P(x)$ and $Q(x)$ be real polynomials, let the real parts of the zeros of $P(x)$ lie on the interval $[a, A]$ and let all the real zeros of $Q(x)$ lie on the interval $[b, B]$. Then the set of real zeros of the derivatives of the rational functions $P(x)/Q(x)$ is proved to be everywhere dense on the four intervals $(-\infty, \min(a, b))$, $(\max(A, B), +\infty)$, (a, A), (b, B). The author refers to N. Obreschkoff [C. R. Acad. Bulgare Sci. Math. Nat. 1, no. 1, 5–8 (1948); these Rev. 11, 102] and D. Markovitch [Bull. Soc. Math. Phys. Serbie 4, no. 3–4, 1–5 (1952); these Rev. 14, 965]. *M. Marden* (Milwaukee, Wis.).

Weinberg, Louis. Test for zeros in the unit circle. J. Appl. Phys. 24, 1251–1252 (1953).

This paper contains a description of the continued fraction method (which is summarized in Marden, The geometry of the zeros of a polynomial in a complex variable [Amer. Math. Soc., New York, 1949; these Rev. 11, 101]), for the determination of stability, that is, for the determination of the location of the zeros of the characteristic polynomial in the left-half plane or in the unit circle. Other criteria are also indicated. *E. Frank* (Chicago, Ill.).

Nassif, M. On the mode of increase of simple sets of polynomials of given zeros. Proc. Math. Phys. Soc. Egypt **4** (1952), no. 4, 29–36 (1953).

The author considers the set of polynomials defined by $P_{mk+i}(z)=(z+a_i)^{mk+i}$, $i=0, 1, \cdots, k-1$; $m=0, 1, 2, \cdots$. He finds that the set is of order 1, type $1/|\rho|$ (in the sense of J. M. Whittaker), where ρ is the root of smallest absolute value of a certain determinant. In particular, the set $\{z+(-1)^n\}^n$, which was used by Eweida [Duke Math. J. **14**, 865–875 (1947); these Rev. **9**, 276] as an example, has type $4/\pi$, while the set $(z+\omega^n)^n$, where ω is an imaginary cube root of 1, is of type about 1.339. The results of this paper were applied by the author in another paper [ibid. **19**, 107–113 (1952); these Rev. **13**, 647].

R. P. Boas, Jr. (Evanston, Ill.).

Merli, Luigi. Il fenomeno di Gibbs nell'interpolazione delle funzioni discontinue. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 18–21, 194–197 (1953).

L'auteur généralise son résultat sur l'interpolation des fonctions continues dans le segment $(-1, +1)$, sauf peut-être à l'origine qui peut être un point de discontinuité de première espèce [Boll. Un. Mat. Ital. (3) **4**, 140–146 (1949); ces Rev. **11**, 356], au cas où les noeuds, au lieu d'être les zéros des polynomes de Tchebycheff, sont ceux des polynomes ultrasphériques de Gegenbauer $P_n^{\lambda}(n)$ $(0<\lambda<\frac{1}{2})$.

Pour la fonction: sgn x (sgn $0=0$), les polynomes d'approximation $H_r(x)=y$ sont impairs et ≥ 0 pour $x>0$, leurs arches, dont le nombre augmente indéfiniment avec r, viennent s'aplatir sur le segment $0<y\leq 1$.

J. Favard (Grenoble).

Bernštein, S. N. The definition and fundamental properties of quasi-algebroid and algebroid functions. Doklady Akad. Nauk SSSR (N.S.) **79**, 377–380 (1951). (Russian)

The author calls $f(x)$ quasi-algebroid on $(-1, 1)$ if there is a polynomial $P(x)$ such that (A) $E_n[f(x)P(x)]/E_n[f(x)]\to 0$, where E_n denotes the best approximation by polynomials of degree not exceeding n. If such polynomials $P(x)$ exist, there is a unique one, $D_k(x)$ (with leading coefficient 1), of lowest degree; it is called the characteristic polynomial of $f(x)$ and its roots x_j are called characteristic points. Theorem 1. If $R\geq|x_j+(x_j^2-1)^{\frac{1}{2}}|\geq r$ for all x_j then $R^{-1}\leq\limsup\{E_n[f(x)]\}^{1/n}\leq r^{-1}\leq 1$; hence if all x_j are on $(-1, 1)$, $f(x)$ is not analytic on $(-1, 1)$, while if all the x_j are outside $(-1, 1)$, $f(x)$ is analytic in an ellipse, with foci ± 1, whose semiaxes have sum $r>1$. Now suppose that there is a polynomial $R(x)$ for which

$$E_n[f(x)R(x)]/E_n[f(x)]<\alpha^n \quad (0<\alpha<1);$$

the author calls $f(x)$ asymptotically rational. Theorem 2. A function $f(x)$ is asymptotically rational if and only if it has the form $f(x)=F(x)/R(x)=Q(x)/R(x)+F_1(x)$, where $R(x)$ is a polynomial with no roots in an ellipse whose semiaxes have sum $R>1$, $Q(x)$ is a polynomial, and $F(x)$ and $F_1(x)$ are regular outside an ellipse whose semiaxes have sum $R_1>R$.

The author next introduces what he calls the class of algebroid functions, intermediate between the two classes just defined, having the property that for every polynomial $P(X)$ either

$$E_n[f(x)P(x)]=o\{E_n[f(x)]\} \text{ or } E_n[f(x)]=O\{E_n[f(x)P(x)]\}.$$

If $f(x)$ is algebroid, then (A) is satisfied with $P(x)$ if and only if $P(x)$ is divisible by $D_k(x)$. An algebroid function is called elementary if it has only one characteristic point. Theorem 3. If $f(x)$ is algebroid and has a characteristic polynomial of degree k, it is the sum of at most k elementary functions.

R. P. Boas, Jr. (Evanston, Ill.).

Loster, C. Une propriété des suites de polynômes homogènes de deux variables complexes bornées sur une courbe. Ann. Soc. Polon. Math. **25** (1952), 210–217 (1953).

Let (1) $\{P_n(x, y)=\sum_{j=0}^{n}a_{n-j,j}x^{n-j}y^j\}$ be a sequence of polynomials, with $a_{n-j,j}$, x, y complex, and let (2) C: $x=x(t)$, $y=y(t)$, $0\leq t\leq 1$, be a curve (x, y continuous complex functions), such that every subarc is of positive écart in the sense of Leja [Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1933**, 453–461 (1934)]. It is shown that if (1) is bounded at every point of C, then to every $\epsilon>0$ and every point $p_0(x_0, y_0)\in C$ (with x_0, y_0 not both zero) there corresponds a neighborhood $V=V(\epsilon, p_0)$ of p_0 throughout which

$$\limsup_{n\to\infty}|P_n(x, y)|^{1/n}<1+\epsilon. \tag{3}$$

This result was stated by Leja, who proved the simpler case of one complex variable [Math. Ann. **108**, 517–524 (1933)].

I. M. Sheffer (State College, Pa.).

***Merli, Luigi. Sopra alcune disuguaglianze riguardanti i polinomi ultrasferici di Jacobi.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 151–155. Casa Editrice Perrella, Roma, 1953.

Denoting by $F_n^{(\lambda)}(x)$ the ultraspherical polynomials normalized by the condition $F_n^{(\lambda)}(1)=1$, the author proves the inequality

$$[F_n^{(\lambda)}(x)]^2-kF_{n-1}^{(\lambda)}(x)F_{n+1}^{(\lambda)}(x)>0$$

provided $-1<x<1$, $1<\lambda<2$ and

$$k\leq n(n+\lambda+1)/[n(n+\lambda+1)+\lambda].$$

This is a natural generalization of a result of Colucci [Boll. Un. Mat. Ital. (3) **5**, 289–292 (1950); these Rev. **12**, 607].

G. Szegő (Stanford, Calif.).

***Zweiling, Klaus. Grundlagen einer Theorie der biharmonischen Polynome.** Verlag Technik, Berlin, 1952. viii+130 pp. (5 plates, 1 insert). DM 18.50.

The biharmonic polynomials investigated in this book are

$$P_{0,0}=1, \quad P_{1,0}=x, \quad P_{1,1}=xy, \quad P_{2n,0}=\tfrac{1}{2}x(w^{2n-1}+\bar{w}^{2n-1}),$$

$$P_{2n,1}=\frac{x(w^{2n}-\bar{w}^{2n})}{4ni},$$

$$P_{2n+1,0}=\frac{1}{4n}[2nx(w^{2n}+\bar{w}^{2n})-iy(w^{2n}-\bar{w}^{2n})],$$

$$P_{2n+1,1}=-\frac{y(w^{2n+1}+\bar{w}^{2n+1})+i(wn+1)x(w^{2n+1}-\bar{w}^{2n+1})}{4n(2n+2)},$$

and the polynomials $P_{0,n}$ and $P_{1,n}$ obtained from these by interchanging x and y: here $w=x+iy$ and $\bar{w}=x-iy$.

The author proves that every polynomial solution of $\Delta\Delta z=0$ (Δ is Laplace's operator) is a linear combination of his biharmonic polynomials. He obtains explicit representations in both Cartesian and polar coordinates, recurrence relations, differentiation and integration formulas, and the nodes (zeros) of these polynomials. Auxiliary functions

$$p_{2n,0}=x^{-1}P_{2n,0}, \quad p_{2n,1}=x^{-1}y^{-1}P_{2n,1},$$

$$p_{2n+1,0}=x^{-1}P_{2n+1,0}, \quad p_{2n+1,1}=x^{-1}y^{-1}P_{2n+1,1}$$

are also introduced, the symmetry properties are explained, and the application of biharmonic polynomials in problems concerning bending of rectangular elastic plates is indicated.

The book contains axonometric representations of the surfaces $z=P_{i,j}(x, y)$ for $i=0(1)12$, $j=0, 1, 0\leqq x, y\leqq 1$, the explicit representation of these polynomials and of their nodes, numerical tables of $P_{i,j}(x, y)$ for $i=0(1)12$, $j=0, 1$, $x, y=0(.1)1$, and of $p_{11,1}$, $p_{12,0}$, $p_{12,1}$ and an inset which expresses $\int P_{m,n}P_{i,k}dx$ as a combination of products of biharmonic polynomials, for $m+n\leqq 6$, $i+k\leqq 6$.

A. Erdélyi (Pasadena, Calif.).

Special Functions

Olver, F. W. J. Note on the asymptotic expansion of generalized hypergeometric functions. J. London Math. Soc. **28**, 462–464 (1953).

The author points out an error in a result of Barnes [Proc. London Math. Soc. (2) **5**, 59–116 (1907)]. The statement "$m=0, 1, \cdots, \mu$" on the lower half of p. 111 should be corrected to "$m=0, 1, \cdots, \mu$ if μ is even, $m=1/2, 3/2, \cdots, \mu-\frac{1}{2}$ if μ is odd". *A. Erdélyi* (Pasadena, Calif.).

Koschmieder, Lothar. Integrals with hypergeometric integrands. II. Univ. Nac. Tucumán. Revista A. **9**, 63–78 (1952). (Spanish)

Continuing an earlier paper [Acta Math. **79**, 241–254 (1947); these Rev. **9**, 351; **11**, 870] on hypergeometric functions of one and two variables, the author now gives similar results for hypergeometric functions of n variables.

A. Erdélyi (Pasadena, Calif.).

Slater, L. J. Two double hypergeometric integrals. Quart. J. Math., Oxford Ser. (2) **4**, 127–131 (1953).

A very general double integral of the Barnes type representing hypergeometric double series of unit argument is evaluated by the calculus of residues in two different ways, thus leading to a transformation formula for hypergeometric double series. The corresponding formula for basic double series is similarly obtained. *A. Erdélyi.*

Ved Prakash. On certain integrals involving hypergeometric functions and operational images. Ganita **2**, 68–74 (1951).

The object of the present note is to obtain certain integrals involving hypergeometric functions and a few operational images which are believed to be new. (Author's introduction.) *A. Erdélyi* (Pasadena, Calif.).

Bose, B. N. On certain integrals involving hypergeometric functions. Bull. Calcutta Math. Soc. **44**, 169–174 (1952).

In this paper the integral

$$\int_0^\infty f(z)z^\rho(z+k)^{-\sigma-1}dz$$

is evaluated when $f(z)$ is ${}_1F_2(a; b, c; -z)$, ${}_1F_2(a; b, c; -z^2)$, or ${}_1F_1(a; b; -z)$. There are a number of special cases, and in section 5 operational calculus is used to derive new integrals from those obtained in the first four sections.

A. Erdélyi (Pasadena, Calif.).

Agarwal, R. P. On the partial sums of series of hypergeometric type. Proc. Cambridge Philos. Soc. **49**, 441–445 (1953).

The principal result of this paper is the transformation of a truncated particular basic hypergeometric series ${}_8\Phi_7$ into a constant multiple of another truncated ${}_8\Phi_7$. As a corollary, one obtains a similar result for particular truncated ordinary (generalized) hypergeometric series ${}_7F_6$. A number of particular cases follow, including similar transformations of ${}_{p+1}\Phi_p$ for the expression of a truncated ${}_2\Phi_2$ in terms of an infinite ${}_8\Phi_7$, the sum of a truncated Saalschützian ${}_2\Phi_1$ in closed form, etc. *A. Erdélyi* (Pasadena, Calif.).

Agarwal, R. P. Some transformations of well-poised basic hypergeometric series of the type ${}_8\Phi_7$. Proc. Amer. Math. Soc. **4**, 678–685 (1953).

Following Whipple's classification of well-poised ${}_7F_6$, the author classifies the well-poised ${}_8\Phi_7$ series, and studies systematically the two-term and three-term relations existing between the 192 allied series which occur in his classification.

A. Erdélyi (Pasadena, Calif.).

Bromberg, J. New representations of Whittaker's confluent hypergeometric function. Revista Unión Mat. Argentina **15**, 157–172 (1953).

The author derives a known integral representation of $M_{k,m}(x)$ and, by expansion of the integrand, obtains an expansion of this function in terms of modified Bessel and modified Struve functions. *A. Erdélyi.*

Srivastava, H. M., and Chak, A. M. Mitra-Srivastava's $P_{n,k}(x)$ function and confluent hypergeometric function. Ganita **3**, 19–22 (1952).

The $P_{n,k}$ function is a disguised $W_{k,m}$ function. In the present note the authors transcribe some known relations.

A. Erdélyi (Pasadena, Calif.).

Ragab, Fouad M. Recurrence formulae for the E-functions. Proc. Math. Phys. Soc. Egypt **4** (1952), no. 4, 127–136 (1953).

The author gives two finite expansions for $E(p; \alpha_r: q; \rho_s: x)$. One of these is of the form

$$\sum_{m=0}^{n} A_m x^{-m} E(p+1; \alpha_r+m, 1+\alpha+n+m: \rho_s+m, 1+\alpha+2m: x)$$

and the other of the form

$$\sum_{m=0}^{n} B_m x^{-m} E(p+2; \alpha_r+m, 1+\alpha+n+m, 1+\alpha-\beta+m: q+2; \rho_s+m, 1+\alpha-\beta+n+m, 1+\alpha+2m: x),$$

the coefficients A_m and B_m being certain combinations of gamma functions. Two proofs are given for each of these expansions, one by induction, and one using the Barnes integral representation of the E-function. *A. Erdélyi.*

Erdélyi, Arthur. Funzioni epicicloidali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 393–394 (1953).

The solution of the wave equation $u_{xx}+u_{yy}+\lambda^2 u=0$ in epicycloidal coordinates defined by $x+iy=na\rho e^{\phi i}+b\rho^n e^{n\phi i}$, was discussed by Agostinelli [same Rend. (8) **7**, 316–320 (1950); **11**, 339–344 (1951); these Rev. **11**, 440; **13**, 843] who introduced what he called epicycloidal functions. Erdélyi now shows that these functions can be expressed very simply, in finite terms, by means of Bessel functions.

E. T. Copson (St. Andrews).

Gray, Marion C. Legendre functions of fractional order. Quart. Appl. Math. **11**, 311–318 (1953).

Legendre functions of fractional order occur in several physical problems, and the author gives some results which

were developed a few years ago in connection with Schelkunoff's antenna theory. Since $P_{-\nu-1}=P_\nu$, and $P_{\nu+n}$ may be computed by means of recurrence relations, it is sufficient to discuss P_ν for $-\frac{1}{2}<\nu<\frac{1}{2}$. When $\nu=\pm\frac{1}{2}$, Legendre functions can be expressed in terms of complete elliptic integrals.

The expansion of $P_\nu(\cos\theta)$ in powers of $\sin\frac{1}{2}\theta$ is quite rapidly convergent for small θ, but it fails to exhibit the analytic nature of P_ν as a function of ν. For this reason the author expands $P_\nu(\cos\theta)$ in powers of ν: the series can be used conveniently to compute $P_\nu(\cos\theta)$ when $0<\theta<\pi/2$ and $-\frac{1}{2}<\nu<\frac{1}{2}$. For the neighborhood of $\theta=\pi/2$ special formulas are developed, there is an expansion for $\pi/2<\theta<\pi$, and in the neighborhood of $\theta=\pi$ an expansion of Hille's is used.

The author discusses the zeros of $P_\nu(\cos\theta)$ for fixed θ and variable ν, and gives a 6D table of $P_\nu(\cos\theta)$ for $\nu=.1(.1)2$, $\theta=10°(10°)170°, 175°$. *A. Erdélyi.*

Palamà, G. Sul Wronskiano delle funzioni di Laguerre di 1ª e 2ª specie e su dei polinomi ad esse associati. Boll. Un. Mat. Ital. (3) **8**, 185–193 (1953).

The author computes the Wronskian of the Laguerre polynomial $L_n^\alpha(x)$ and the corresponding function of the second kind $l_n^\alpha(x)$ [defined in an earlier paper, same Boll. (3) **5**, 72–77 (1950); these Rev. **12**, 25], and for $\alpha=\pm1/2$ obtains the corresponding result for Hermite polynomials. He also shows that $l_n^\alpha(x)$ may be expressed in terms of $L_n^\alpha(x)$, $l_0^\alpha(x)$, and a certain auxiliary polynomial of degree $n-1$, and obtains some formulas involving these polynomials. He concludes with formulas relating to $l^\alpha_{-n-1}(x)$. *A. Erdélyi* (Pasadena, Calif.).

Riekstyn'š, Ē. Ya. On special functions applicable to the solution of telegraph equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 125–132 (1953). (Russian)

The functions in question are the real and imaginary parts of

$$\sum_{k=0}^{\infty}\left(\frac{x}{y}\right)^{(\nu+2k)/2} I_{\nu+2k}(2(xy)^{\frac{1}{2}})\exp\{i(2k+\nu+\epsilon)\varphi\},$$

for real $x, y\geqq0$ and any real ν, φ, ϵ; their use is recommended for the solution of telegraphic problems in which the author's previously introduced functions [same journal **15**, 485–494 (1951); these Rev. **13**, 344] would have had to be used with complex arguments. After listing numerous results for these functions they are used to express the transient term in the solution of $-u_x=Ri+Li_t$, $-i_x=Gu+Cu_t$, with $u(x,0)=i(x,0)=0$, $u(0,t)=E\sin\delta t$. *F. V. Atkinson.*

van der Corput, J. G. On the integral from zero to infinity of the power of e with exponent $-pe^x+qe^{rx}+sx$. Nieuw Arch. Wiskunde (3) **1**, 99–104 (1953).

This paper contains some results on

$$G(p,q,r,s)=\int_0^\infty \exp(-pe^x+qe^{rx}+sx)dx.$$

First the author expands $G(p,0,r,s)-p^{-s}\Gamma(s)$ in powers of p, and gives the modification necessary when $s=0, -1, -2, \cdots$. Then he gives the expansion in powers of q of $G(p,q,r,s)$ together with the modification necessary when $s+kr=0, -1, -2, \cdots$ for some non-negative integer k. When $r=-1$ and s is not an integer, the expansion simplifies and becomes a series of Bessel functions. The paper concludes with the elegant identity

$$G(p,-p,-1,s)+G(p,-p,-1,-s)=2K_s(2p)$$

where K is the modified Bessel function of third kind. *A. Erdélyi* (Pasadena, Calif.).

Dingle, R. B. The anomalous skin effect and the reflectivity of metals. Evaluation of the integrals appearing in the expressions for the surface impedance. Appl. Sci. Research B. **3**, 69–99 (1953).

The author gives expansions in both ascending and descending powers of x for the integrals

$$\frac{2}{\pi}\int_0^\infty \frac{dt}{t^2+xf(t)},\quad \frac{1}{\pi}\int_0^\infty \ln\left(1+\frac{xf(t)}{t^2}\right)dt$$

where $f(t)=2t^{-3}[(1+t^2)\tan^{-1}t-t]$. The Mellin transformation (with respect to x) is used for the formal computation of the coefficient in these expansions. The appendices contain the justifications of the expansions. *A. Erdélyi.*

Fassò, Costantino. Di un integrale intervenuto in una questione di idraulica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 471–497 (1951).

The author studies in detail the integral

$$(1)\qquad y=\int_1^z \frac{t^3dt}{z+t^3}=\Phi(x,z)-\Phi(1,z),$$

discusses the level curves $z=$const. of the surface represented by (1) in (x,y,z) space and gives a few numerical values of $\Phi(x,z)$, where $\Phi(0,z)=0$. *A. Erdélyi.*

Arató, Mátyás, et Freud, Géza. Le calcul des intégrales d'interaction à un seul centre modifié. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 369–375 (1953). (Hungarian. Russian and French summaries)

Let O be a fixed point, and let P_1, P_2 be variable points, in space. Denoting by r_1, r_2, r_{12} the distances OP_1, OP_2, P_1P_2 respectively, and by $d\tau_1$, $d\tau_2$ the elements of volume as P_1, P_2 range over the entire space, the author computes the integrals

$$I_{mn}(r_0)=\int\int \frac{r_1^m r_2^n}{r_{12}+r_0}\exp(-r_1-r_2)d\tau_1 d\tau_2,\quad m, n \text{ integers}.$$

He gives explicit expressions in terms of elementary functions and integrals of the form

$$\int_0^{r_0} x^n e^{-x}\log x\,dx$$

for $0\leqq n\leqq m\leqq2$, and approximations for small r_0 for $0\leqq n\leqq m\leqq3$. The approximations contain only elementary functions, they are stated to hold for $0<r_0<10^{-4}$, and estimates for the error (usually about the order of 10^{-16}) are given. *A. Erdélyi* (Pasadena, Calif.).

Harmonic Functions, Potential Theory

***Spencer, D. C. Real and complex operators on manifolds.** Contributions to the theory of Riemann surfaces, pp. 203–227. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The theory of Green's functions has been extended to a compact Riemannian manifold M by de Rham [G. de Rham and K. Kodaira, Harmonic integrals, Institute for Advanced Study, Princeton, 1950; these Rev. **12**, 279] as follows: there exists a unique self-adjoint operator G on M such that, if ϕ is a regular p-form on M and $H\phi$ is its har-

monic part, $\Delta G\phi=\phi-H\phi=G\Delta\phi$, $HG\phi=0$, where Δ is the Laplace-Beltrami operator.

A finite Riemannian manifold B is a closed domain of a Riemannian space which can be isometrically embedded as a closed subdomain with a smooth boundary in a compact Riemannian manifold M. The present paper extends the theory of Green's operators to finite Riemannian manifolds. In this case we have to consider three operators H, F, E in place of the single operator H. H is the space of harmonic p-forms, F the space of p-forms which satisfy $d\phi=0=\delta\phi$, and E the space of p-forms which are harmonic on M and vanish outside B. We also denote by H, F, E the operations of projections of p-forms on these spaces. It is proved that there exists a uniquely defined pair of adjoint operators G, G', and two uniquely defined self-adjoint operators N and Γ such that

$$\begin{array}{lll}\Delta G\phi=\phi-H\phi, & G'\Delta\phi=\phi-H\phi, & EG=0,\\ \Delta N\phi=\phi-F\phi, & N\Delta\phi=\phi-F\phi-p_n\phi, & FN=0,\\ \Delta\Gamma\phi=\phi-E\phi, & \Gamma\Delta\phi=\phi-E\phi-p_\Gamma\phi, & E\Gamma=0,\end{array}$$

where p_n, p_Γ are certain boundary operators associated with N and Γ.

Some applications of these formulae are given.

When the manifold under consideration is a complex manifold with a Kähler metric, it is known [e.g., Hodge, Proc. Cambridge Philos. Soc. 47, 504–517 (1951); these Rev. 13, 75] that the various operators d, δ, Δ split up into pairs of conjugate operators which have a considerable degree of independence, and there is a corresponding splitting of the Green's operators. The effect of this when the foregoing analysis is applied to a Kähler manifold is examined, and as an application a generalisation is made to (ρ, σ)-forms of the classical formula

$$\phi(z)=\frac{1}{2\pi i}\int_{CB}\phi(\zeta)\frac{d\zeta}{\zeta-z}.$$

W. V. D. Hodge (Cambridge, England).

Deny, Jacques. Le balayage. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 47–61 (1952).

In this paper the author determines all the kernels which yield a potential theory in E^m for which there is a "sweeping out" process analogous to the "sweeping out" process in Newtonian potential theory; for example, the α kernels due to M. Riesz and Frostmann have that property.

To achieve his purpose, the author must modify some of the usual definitions [Deny, Acta Math. 82, 107–183 (1950); these Rev. 12, 98]. For example, now a "sweeping out" process is possible for the kernel K if for each compact C and each positive measure μ (with compact support), there exists at least one positive measure μ' carried by C such that (1) $K*\mu'\leqq K*\mu$, in E^m, (2) $K*\mu'=K*\mu$ on C, and (3) $\int d\mu'\leqq\int d\mu$. The author shows that (essentially) the set of kernels for which the "sweeping out" process is possible is closed in the "vague" topology. Also, a "sweeping out" is possible for all members of the class $[K_\tau]$, where $[K_\tau]$ consists of all measures of the form $K=a^{-1}(\epsilon+\sigma+\sigma^2+\cdots)$, where a is a positive constant, ϵ is the unit mass at the origin, where σ is a positive symmetric measure of total mass less than one, and where σ^p is the pth iterate of σ by composition. Finally, the author proves that each kernel K of positive type, satisfying two subsidiary conditions, and for which a "sweeping out" process is possible belongs to the "vague" closure of $[K_\tau]$. *M. O. Reade.*

Amanov, T. I. On the solution of a biharmonic problem. Doklady Akad. Nauk SSSR (N.S.) 88, 389–392 (1953). (Russian)

The biharmonic equation

$$\Delta^2u=\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)^2u=0,$$

appearing as Euler's equation for minimization of the functional

$$D_2[u]=\int_\sigma\int\left[u^2_{\rho\rho}+2\left(\frac{1}{\rho}u_{\rho\theta}-\frac{1}{\rho^2}u_\theta\right)^2+\left(\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}\right)^2\right]\rho d\rho d\theta,$$

is here studied for the case of σ the unit circle with boundary conditions $u|_{\rho=1}=\varphi$ and $u_\rho|_{\rho=1}=\psi$. Theorem: Let f be a real-valued function on σ having partial derivatives through the second order and satisfying

$$D[f]=\int_\sigma\int f^2<\infty,$$

$$D_1[f]=\int_\sigma\int\left[\left(\frac{\partial f}{\partial\rho}\right)^2+\left(\frac{1}{\rho}\frac{\partial f}{\partial\theta}\right)^2\right]\rho d\rho d\theta<\infty,$$

$$D_2[f]<\infty;$$

then for almost all θ

(a) $\lim_{\rho\to1}f(\rho e^{i\theta})=\varphi(\theta)$ and $\int_0^{2\pi}\varphi^2<\infty$,

(b) $\lim_{\rho\to1}f_\rho(\rho e^{i\theta})=\psi(\theta)$ and $\int_0^{2\pi}\psi^2<\infty$.

The theorem remains valid for functions f having generalized partial derivatives through the second order in the sense of Nikol'skiĭ [Trudy Mat. Inst. Steklov. 38, 244–278 (1951), p. 256; these Rev. 14, 32]. Using Fourier series methods, the author develops properties of $D_2[u]$ for a restricted class of boundary functions [the class $H_2^{(\nu)}(M)$ introduced by Nikol'skiĭ in the work cited above]. *M. G. Arsove.*

Amanov, T. I. Generalization of a result of S. M. Nikol'skiĭ. Doklady Akad. Nauk SSSR (N.S.) 90, 949–952 (1953). (Russian)

A real-valued function $f(x, y)$ on the open unit disc σ will be said to belong to $W_2^{(m)}(\sigma)$ if it has partial derivatives of orders 1 through $m(\geqq1)$ and

$$D_j[f]=\int_\sigma\int\sum_{\alpha+\beta=j}\frac{j!}{\alpha!\beta!}\left(\frac{\partial^jf}{\partial x^\alpha\partial x^\beta}\right)^2dxdy<\infty\quad(j=0,1,\cdots,m).$$

Theorem: If U [$\in W_2^{(m)}(\sigma)$] is harmonic, then

(i) $$\varphi(\theta)=\lim_{\rho\to1-}U(\rho e^{i\theta})$$

exists a.e. as a function in $L^2(0, 2\pi)$,

(ii) $$U(\rho e^{i\theta})=\tfrac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos k\theta+b_k\sin k\theta)\rho^k,$$

where a_k, b_k are the Fourier coefficients for φ, and (iii) there exists a positive constant C_1 (independent of U) such that $\sum_{k=m}^{\infty}(a_k^2+b_k^2)k^{2m-1}\leqq C_1D_m[U]$. Conversely, if $\varphi\in L^2(0, 2\pi)$ and $\sum_{k=m}^{\infty}(a_k^2+b_k^2)k^{2m-1}<\infty$, then the harmonic function U defined in (ii) belongs to $W_2^{(m)}(\sigma)$ and there exists a positive constant C_2 (independent of U) such that

$$C_2D_m[U]\leqq\sum_{k=m}^{\infty}(a_k^2+b_k^2)k^{2m-1}.$$

The case of $m=1$ is due to Nikol'skiĭ [same Doklady (N.S.) **83**, 23–25 (1952); erratum **84**, 652 (1952); these Rev. **13**, 943]. *M. G. Arsove* (Seattle, Wash.).

Green, John W. On the spherical means of α-potentials. Univ. Nac. Tucumán. Revista A. **9**, 7–11 (1952).

Soit μ une distribution positive dans R^3; posons

$$U_\alpha^\mu(P)=\int PQ^{-\alpha}d\mu(Q) \quad (0<\alpha<3);$$

soit $\mathfrak{M}(U_\alpha^\mu, P, r)$ la valeur moyenne de U_α^μ sur la boule $PQ\leqq r$. Frostman a montré [Dissertation, Lund, 1935] que $A(\alpha)=\sup_\mu \mathfrak{M}(U_\alpha^\mu, P, r)/U_\alpha^\mu(P)$ est fini, et donné la majoration $A(\alpha)\leqq 3/(3-\alpha)$ pour $1<\alpha<3$ (pour $\alpha\leqq 1, A(\alpha)=1$). L'auteur détermine une minoration:

$$A(\alpha)\geqq 3\cdot 2^{1-\alpha}/(3-\alpha)(4-\alpha) \quad (1\leqq\alpha<3)$$

qui est la valeur exacte de A pour $\alpha=1$; il montre d'autre part que $A(\alpha)$ est équivalent à la valeur de Frostman $3/(3-\alpha)$ lorsque $\alpha\to 3$; il traîte également le cas des moyennes superficielles, qui n'a de sens que pour $\alpha<2$: la constante analogue à $A(\alpha)$ est plus simple à déterminer; elle vaut exactement $2^{1-\alpha}/(2-\alpha)$ $(1\leqq\alpha<2)$. *J. Deny.*

***Weinstein, Alexander. The method of singularities in the physical and in the hodograph plane.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 167–178. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

Dans ce mémoire, l'auteur nous offre une revue des nombreuses applications de la théorie généralisée des potentiels de révolution qu'il a développée depuis quelques années [Trans. Amer. Math. Soc. **63**, 342–354 (1948); ces Rev. **9**, 584]. Certains problèmes physiques peuvent s'interpréter comme écoulements de révolution dans un espace dont le nombre de dimensions peut être différent d'un nombre entier; ainsi le problème de la torsion des barres correspond à un écoulement dans un espace à 5 dimensions, et l'équation de Tricomi à un écoulement dans un espace à $2\frac{1}{2}$ dimensions. Une correspondance fondamentale entre la fonction de courant dans un espace à n dimensions et le potentiel dans un espace à $n+2$ dimensions, qui sans doute peut être rattachée aux identités de Darboux pour l'équation d'Euler-Poisson, facilite le calcul des solutions. L'auteur donne diverses expressions analytiques de la solution fondamentale (source) et des fonctions singulières que l'on peut en déduire (doublets). Des applications sont faites à certains problèmes d'hydrodynamique, considérés comme problèmes d'électrostatique généralisée, et à certains problèmes d'écoulements transsoniques lorsque l'approximation de Tricomi est utilisée.

P. Germain (Providence, R. I.).

Jost, Res, and Kohn, Walter. On the relation between phase shift energy levels and the potential. Danske Vid. Selsk. Mat.-Fys. Medd. **27**, no. 9, 19 pp. (1953).

The method of Gelfand and Levitan [Doklady Akad. Nauk SSSR (N.S.) **77**, 557–560 (1951); these Rev. **13**, 240] "is adapted to the problem of determining a central potential from the 'spectral function' corresponding to a given angular momentum".

N. Levinson (Cambridge, Mass.).

Daboni, Luciano. Applicazione al caso del cubo di un metodo per il calcolo per eccesso e per difetto della capacità elettrostatica di un conduttore. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 461–466 (1953).

Let Γ be any closed surface. Its capacity $C(\Gamma)$ (with respect to the infinitely large sphere) can be estimated by means of the following inequalities:

$$(*) \qquad \frac{e}{\max u}\leqq C(\Gamma)\leqq\frac{e}{\min u},$$

where e is the total charge of certain masses taken arbitrarily in the interior of Γ, u is the potential of these masses and the maximum and minimum refer to Γ. [These inequalities are given in Pólya and Szegö, Isoperimetric inequalities in mathematical physics, Princeton, 1951, p. 61; these Rev. **13**, 270.] The method used by the author is illustrated in the special case of a cube Γ. Assuming the edge to be 1, the author defines a distribution on an interior cube of edge 0.98; it consists of a uniform surface distribution along the faces, of a uniform linear distribution along the edges and of equal discrete masses in the vertices. The densities and the point charges are determined by a sort of Ritz method. The main task is the computation of the maximum and minimum involved in the inequalities (*). Accepting the numerical values printed in the paper, the bounds

$$0.654<C(\Gamma)<0.676$$

follow which are much better than those given in the cited book of Polya and Szegö (pp. 76–78). *G. Szegö.*

Differential Equations

Carleson, Lennart. On infinite differential equations with constant coefficients. I. Math. Scand. **1**, 31–38 (1953).

The equation (1) $L[y]\equiv\sum_0^\infty a_n y^{(n)}(x)=0$ is examined principally for non-analytic solutions. It is assumed that $F(z)=\sum_0^\infty a_n z^n$ $(a_0=1)$ is an entire function of at most order one, convergence type, so that $F(z)=\prod_1^\infty(1-z/\lambda_n)^{\mu_n}$ with $\sum_1^\infty \mu_n/|\lambda_n|$ convergent. Set

$$F^*(z)=\prod_1^\infty\left(1+\frac{z}{|\lambda_n|}\right)^{\mu_n}=\sum_0^\infty A_n z^n.$$

A function $y(x)$ is called a solution of equation (1) on the interval (a, b) if (1) holds for $a<x<b$ and if, moreover, (2) $\sum_0^\infty A_n|y^{(n)}(x)|$ converges uniformly on $a+\epsilon<x<b-\epsilon$ for every $\epsilon>0$. The class of such solutions $\{y(x)\}$ is denoted by $C=C(a, b)$. Let $a<0<b$ (which is not an essential restriction) and suppose $y(x)\in C(a, b)$. For each ξ in $a<\xi<b$ set $b_p=b_p(\xi)=\sum_{n=0}^\infty a_{n+p}y^{(n)}(\xi)$ $(p=1, 2, \cdots)$ and

$$G(z;\xi)=\sum_{p=1}^\infty b_p z^{p-1}$$

(in particular, set $G(z)=G(z;0)$). Here $G(z;\xi)$ is entire, either of order <1 or of order 1 and minimal type. To $y(x)$ associate the formal expansion

$$(*) \qquad y(x)\sim\sum_{n=1}^\infty\sum_{k=0}^{\mu_n-1}c_{n,k}x^k e^{\lambda_n x}=\sum_{n=1}^\infty P_n(x)e^{\lambda_n x},$$

where the $c_{n,k}$'s are chosen so that

$$\frac{G(z)}{F(z)}-\sum_{k=0}^{\mu_n-1}\frac{c_{n,k}k!}{(z-\lambda_n)^{k+1}}$$

is regular at $z=\lambda_n$. The following results are established. I) If $y(x)$, $z(x) \in C(a, b)$, and if $\{P_n(x)\}$, $\{Q_n(x)\}$ are the corresponding sequences of polynomials determined in (*), then $\alpha y(x)+\beta z(x) \sim \sum_1^\infty \{\alpha P_n(x)+\beta Q_n(x)\} e^{\lambda_n \xi}$, and

$$y(x+\xi) \sim \sum_1^\infty e^{\lambda_n \xi} P_n(x+\xi) e^{\lambda_n \alpha}, \quad a<\xi<b.$$

II) If the development (*) of a solution $y(x)$ vanishes identically then $y(x) \equiv 0$. III) If $y^{(n)}(x_0)=0$ (all $n \geqq 0$) for some point x_0, then $y(x) \equiv 0$. IV) A necessary and sufficient condition that every solution of (1) which is defined on an open interval containing the closed interval (a, b) be analytic on (a, b) is that $\liminf_{n \to \infty} |\alpha_n/\beta_n| > 0$ $(\lambda_n=\alpha_n+i\beta_n)$.

I. M. Sheffer (State College, Pa.).

Haas, Feliz. On the global behavior of differential equations on two-dimensional manifolds. Proc. Amer. Math. Soc. **4**, 630–636 (1953).

Let M be a closed two-dimensional orientable manifold, V a vector field on M defining the differential equation, and $\bar{C}$ the set of points to which the positive semi-characteristic C^+ comes arbitrarily close infinitely often. The author proves the following theorem: If the vector field V satisfies a Lipschitz condition and has at most a denumerable number of singular points, and if, for some C^+, $\bar{C}$ does not contain any singular points, then M is a torus and V is free of singular points or $\bar{C}$ is nowhere dense on M.

C. J. Titus (Ann Arbor, Mich.).

Katô, Tizuko. Sur les points singuliers des équations différentielles ordinaires du premier ordre. II. Nat. Sci. Rep. Ochanomizu Univ. **4**, 36–39 (1953).

[For part I see same Rep. **2**, 13–17 (1951); these Rev. **14**, 274.] Let $P(x, y)$ and $Q(x, y)$ be polynomials in y with coefficients regular in x for $|x|<\Delta$, and set $P_0(y)=P(0, y)$ and $Q_0(y)=Q(0, y)$. Suppose that the equation

$$Q_0(y)dy=P_0(y)dt$$

has a solution $\phi(t)$ that is holomorphic in a horizontal strip in the t-plane and has a real period ω. Sufficient conditions are given in order that certain solutions of the equation $Q(x, y)xdy=P(x, y)dx$ be expressible as $\sum p_k(c+\ln x)x^k$ where the coefficients $p_k(c+t)$ are ω-periodic in t,

$$p_0(c+t)=\phi(c+t),$$

and the constant c is chosen so as to satisfy an initial condition.

F. A. Ficken (Knoxville, Tenn.).

Ascoli, G. Sul comportamento asintotico delle soluzioni dell'equazione $y''-(1+\eta)y=0$. Boll. Un. Mat. Ital. (3) **8**, 115–123 (1953).

The paper contains an account of the asymptotic properties, as $x \to +\infty$, of the differential equation

$$y''-(1+\eta(x))y=0.$$

Most of the results have been proved before, important papers in this connection being the articles by P. Hartman [Trans. Amer. Math. Soc. **63**, 560–580 (1948); these Rev. **9**, 589], N. Levinson [Duke Math. J. **15**, 111–126 (1948); these Rev. **9**, 509], and A. Wintner [ibid. **15**, 55–67 (1948); these Rev. **9**, 509]. The purpose of the present paper is to develop the theory of this problem in a simplified and essentially self-contained fashion. Various hypotheses on $\eta(x)$ are studied, e.g., $|\eta|^p$ $(1 \leqq p \leqq 2)$ integrable from 0 to $+\infty$; or $1+\eta=\beta^2+C$, with $\beta^2 \to 1$, as $x \to \infty$, and of bounded variation, and $|C|$ integrable from 0 to ∞. The proofs make systematic use of conveniently constructed auxiliary functions of y and y'.

W. Wasow (Los Angeles, Calif.).

Gagliardo, Emilio. Sul comportamento asintotico degli integrali dell'equazione differenziale $y''+A(x)y=0$ con $A(x) \geqq 0$. Boll. Un. Mat. Ital. (3) **8**, 177–185 (1953).

Under the assumption that $A(x)$ is continuous for $x \geqq x_0>0$ the author derives several simple criteria for the asymptotic behavior, as $x \to +\infty$, of the solutions of the differential equation in the title. Among these results are the following. (1) If $\int_{x_0}^\infty A(x)dx=\infty$, or if $A(x) \geqq \frac{1}{4}x^{-2}$ and $\int_{x_0}^\infty (xA(x)-\frac{1}{4}x^{-1})dx=\infty$, then all integrals are oscillatory. (2) If $\int_{x_0}^\infty A(x)dx<\infty$, and if a solution possesses the infinitely many zeros $a_1<a_2<\cdots$, then $\sum_{n=1}^\infty (a_{n+1}-a_n)^{-1}<\infty$. (3) If $\int_{x_0}^\infty x^2A(x)dx=\infty$, no solution has a non-horizontal asymptote. If $\int_{x_0}^\infty xA(x)dx=\infty$, no solution has an asymptote. (4) If $\int_{x_0}^\infty x^2A(x)dx<\infty$, every non-oscillatory solution has an asymptote. (Reviewer's note: The author's theorem I was proved by A. Wintner [Amer. J. Math. **73**, 368–380 (1951); these Rev. **13**, 37].)

W. Wasow (Los Angeles, Calif.).

Zubov, V. I. Some sufficient criteria for stability of a nonlinear system of differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 506–508 (1953). (Russian)

Consider the n-vector, $n \times n$ matrix system $\dot{x}=A(x; t)x$ and set $G=\frac{1}{2}(A+A')$. Let $\lambda_i(x; t)$ be the characteristic roots of B. Theorem 1. If the $\lambda_i \leqq 0$, the origin is stable. The proof rests upon the immediate relation in which x is considered as a column vector: $\frac{1}{2}d(x' \cdot x)/dt=x'Bx$. An orthogonal transformation is applied normalizing B. In the new unknowns z_i one finds (euclidean norm)

$$\|z(t)\|^2=\|z(t_0)\|^2 \exp 2\int_{t_0}^t \frac{\sum \lambda z_i^2(t)}{\|z(t)\|^2}dt$$

leading at once to the theorem. Noteworthy consequences: (1) If $\lambda_i<-\delta<0$, then the origin is asymptotically stable. (2) A need not satisfy the conditions for the uniqueness of solutions. Under the circumstances a trajectory may reach the origin in finite time, but under the conditions of the theorem it will not be able to leave the origin. (3) Consider a system $\dot{x}=X(x; t)$, where $X(0; t)=0$. Let $J=(\partial X_n/\partial x_i)$. If the characteristic roots of $\frac{1}{2}(J+J')$ are $\leqq 0$, the origin is asymptotically stable. (4) Let $\lambda(t) \geqq \lambda_i(x; t)$ [$\leqq \lambda_i(x; t)$] for $t_0 \leqq t<+\infty$. If $\int_{t_0}^t \lambda(t)dt<+\infty$ [$=+\infty$], then the origin is stable [unstable].

Let $A(0; \infty)=C$, a constant matrix, and let $\nu_1, \cdots, \nu_n$ be the characteristic roots of C. Theorem 2. If $A(0; \infty)$ is continuous at $x=0$, $t \to +\infty$ and if the ν_i all have negative real parts then the origin is asymptotically stable. Noteworthy consequence: Take again $\dot{x}=X(x; t)$ as before. If the characteristic roots $f_i(x; t)$ of J are such that the $\rho_i(0; \infty)$ have negative real parts then the origin is asymptotically stable for the system.

S. Lefschets.

Reeb, Georges. Sur la stabilité des solutions périodiques de l'équation différentielle $X(x, y)dx+Y(x, y)dy=0$. Collectanea Math. **4**, no. 2, 51–56 (1951).

The principal result is a rather complicated set of sufficient conditions that the equation: $\ddot{x}+f(x)\dot{x}+g(x)=0$ have at most one periodic solution.

W. Kaplan.

Sestini, Giorgio. Criteri di stabilità per il moto di un punto soggetto a forza elastica, a resistenza e ad una forza disturbatrice. Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 559–564. Casa Editrice Perrella, Roma, 1953.

The differential equation $\ddot{y}+\omega^2y\phi(\dot{y})=f(t)$ is investigated as to the stability of its solutions as $t \to +\infty$. Here ω is a positive constant, $\phi(\dot{y})$ is continuous, and $\dot{y}\phi(\dot{y}) \geqq 0$, for

$-\infty<\dot{y}<\infty$. If, in addition, (*) $\int^{\infty}|f(t)|dt<C$ then any solution that is defined in $t_0\leqq t<\infty$ is shown to be stable, in the sense that $|y(t)|$ and $|\dot{y}(t)|$ are bounded. If instead of (*) it is assumed that $\phi(\dot{y})=2\epsilon\dot{y}(t)=R(\dot{y})$ with positive constant ϵ, where $R(\dot{y})\dot{y}\geqq 0$, and $R(\dot{y})$ is uniformly Lipschitzian of order $\alpha<1$ in every finite interval, then stability of one solution implies stability of all solutions defined in $t_0\leqq t<\infty$. *W. Wasow* (Los Angeles, Calif.).

Cremer, L. Die Verringerung der Zahl der Stabilitätskriterien bei Voraussetzung positiver Koeffizienten der charakteristischen Gleichung. Z. Angew. Math. Mech. 33, 221–227 (1953). (English, French and Russian summaries)

This paper is concerned with certain refinements of the theory of the stability of linear systems. It is shown that if the coefficients of the characteristic equation are positive, it suffices, in order to determine whether or not the system is stable, to consider only every other determinant in the well known Hurwitz sequence. A rule of signs is given which determines the number of pairs of conjugate complex roots with positive real parts. *L. A. MacColl.*

Lefévre, Paul. L'étude de la stabilité des systèmes linéaires par la méthode du diagramme de phase généralisé. Mém. Artillerie Française 26, 503–588 (1952).

This is a very elaborate discussion of the stability of linear systems, with special reference to closed-cycle control systems. The fundamental theory presented contains no novelties, but the author insists on a large number of computational devices which are intended to facilitate practical applications of the theory. In general, these devices do not differ substantially from the ones which are in current use. In addition, the author carries through a detailed discussion of a great number of special cases. In the opinion of the reviewer, the inclusion of this wealth of minute detail is unfortunate, for it will be unnecessary to the readers who have a sound understanding of the fundamental theory, and it will be merely baffling to those who have not.

L. A. MacColl (New York, N. Y.).

***van der Pol, Balth. Note sur les propriétés des solutions d'une équation différentielle, que l'on peut déduire directement de l'équation différentielle elle-même. Note on the properties of solutions of a differential equation which may be derived directly from the differential equation itself.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 159–167, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and English)

If an ordinary differential equation possesses a solution of period T many useful relations involving mean values over the period can be derived by multiplication of the differential equation with suitable functions of the dependent variable and its derivatives, followed by an integration over the period. The same technique is often valuable for non-periodic solutions, if the integration is extended from one zero of the solution to another. These procedures are illustrated by numerous examples. *W. Wasow.*

Viswanatham, B. The existence of harmonic vibrations. Proc. Amer. Math. Soc. 4, 371–372 (1953).

If the system $\dot{x}=X(x,t)$ (x an n-vector; X periodic in t) admits a bounded solution each of whose components is a monotonic function, there exists a harmonic vibration.

J. L. Massera (Montevideo).

de Castro, Antonio. Sopra l'equazione differenziale delle oscillazioni non-lineari. Rivista Mat. Univ. Parma 4, 133–143 (1953).

The author considers the differential equation

$$\ddot{x}+f(x,\dot{x},t)\dot{x}+g(x)=e(t),$$

where the functions f and e are periodic with respect to t, with the period T. The existence, uniqueness, and stability of a periodic solution, with period T, are proved under a rather weak set of further hypotheses concerning the functions f, g, and e. *L. A. MacColl* (New York, N. Y.).

***Graffi, Dario. Sur la période d'oscillation des systèmes non linéaires à plusieurs degrés de liberté. Sul periodo delle oscillazioni nei sistemi non-lineari a più gradi di libertà.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 189–193; discussion, p. 194, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and Italian)

The author considers two inductively coupled electric circuits whose currents i_1 and i_2 satisfy the differential system

$$L_1\frac{d^2i_1}{dt^2}+M\frac{d^2i_2}{dt^2}+f_1(i_1)\frac{di_1}{dt}+\frac{i_1}{c_1}=0,$$

$$M\frac{d^2i_1}{dt^2}+L_2\frac{d^2i_2}{dt^2}+f_2(i_2)\frac{di_2}{dt}+\frac{i_2}{c_2}=0,$$

with non-linear resistances $f_1(i_1)$, $f_2(i_2)$. It is proved that if this system possesses a periodic solution then its period is not less than the smaller one of the periods of the corresponding linear system obtained by setting $f_1(i_1)=f_2(i_2)=0$. The author states that the extension of the argument to the case of n degrees of freedom, as well as to circuits with capacitive coupling, does not offer any new difficulties.

W. Wasow (Los Angeles, Calif.).

Seifert, George. On certain solutions of a pendulum-type equation. Quart. Appl. Math. 11, 127–131 (1953).

L'autore indica delle esplicite condizioni per le funzioni $f(\theta)$, $g(\theta)$ (periodiche e di periodo 2π) affinchè l'equazione differenziale

$$\frac{d^2\theta}{dt^2}+f(\theta)\frac{d\theta}{dt}=g(\theta)$$

ammetta una soluzione periodica di seconda specie,

$$\theta'(t+2\pi)=\theta'(t),$$

nell'ipotesi che sia $f(\theta)>0$ e che $g(\theta)=0$ abbia un numero finito di radici nell'intervallo $0\leqq\theta\leqq 2\pi$. *L. Amerio.*

Nishino, Kichiji. Some notes on the sub-harmonic resonance in the non-linear mechanical vibratory system. J. Jap. Soc. Appl. Mech. 3, 121–126 (1950). (Japanese. English summary)

The author deals theoretically with some properties of subharmonic resonance in a vibratory system with a small nonlinear characteristic. At first he treats Duffing's equation with viscous damping and then the case of a restoring force expressed as a curve of nth degree in the displacement.

Author's summary.

Duncan, W. J. Some related oscillation problems. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda 2707 (1949), 12 pp. (1953).

The author observes that a linear equation may remain unchanged when acted upon by certain types of differential

operators, although the boundary conditions for a given problem will generally be changed. In case the new boundary conditions are physically significant one has a simple way of obtaining the solution to this second boundary-value problem from the first. The idea is illustrated by a number of shaft-torsion and beam-flexure problems. *E. Pinney.*

***Minorsky, N. Sur le phénomène Béthenod.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 223–234; discussion, pp. 235–236, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953).

Suppose an iron pendulum is near an electromagnet carrying an alternating current. The magnet exerts a force on the pendulum, and the pendulum affects the self-inductance of the magnet. Under suitable conditions a self-sustained oscillation of the pendulum will develop, which is the "phenomenon" referred to. The system has two degrees of freedom and is described in terms of two nonlinear differential equations in two unknowns. In the discussion at the end of the paper it was noted by Gauthier that stable solutions to such systems may be represented by a point moving on a torus (analogous to a point in the phase plane in the one degree of freedom case) whose dimensions may be determined approximately. The author uses a more heuristic approach. He assumes at the outset that the pendulum oscillates harmonically with unknown period and amplitude. This yields a linear equation for the current in the magnet. Solving this and inserting the result in the original equations, a differential equation of nonlinear Mathieu type is obtained for the deflection angle θ of the pendulum. The question now arises as to the agreement between this equation and the original assumption that θ oscillates harmonically. This is investigated in detail using a modified averaging method (called a "stroboscopic" method because the values of the variables are calculated at uniform intervals, assuming they vary smoothly between). A large number of special cases occur due to the large number of arbitrary constants in the equations. *E. Pinney.*

Minorsky, Nicolas. Sur l'extinction asynchrone. C. R. Acad. Sci. Paris **237**, 643–645 (1953).

The author investigates the equation

$$x''-(a-cx^2)x'+x=\nu\sin\omega t$$

for a, c, ν small, heuristically using his "stroboscopic method", and finds an oscillation of frequency near 2π. He states that this oscillation loses stability for ω sufficiently far from 2π. *E. Pinney* (Berkeley, Calif.).

Colombo, Giuseppe. Sopra un singolare caso che si presenta in un problema di stabilità in meccanica nonlineare. Rend. Sem. Mat. Univ. Padova **22**, 123–133 (1953).

The differential equation

$$\ddot{v}+\alpha(v^2-1)\dot{v}+\omega^2 v=-2\alpha\omega\sin 3\omega t \quad (\alpha>0)$$

admits the solution $v=2\cos\omega t$. Here the author investigates the stability of this solution. By reducing the variational equation to a Hill's equation, and solving the latter equation by a familiar method of successive approximations, he shows that the solution in question is unstable for small values of α. Various delicate points, arising in connection with the use of approximations in the solution of problems concerning stability, are commented on. *L. A. MacColl.*

Cimino, Massimo. Sulle soluzioni dell'equazione generale del potenziale newtoniano di una sfera fluida in equilibrio. Boll. Un. Mat. Ital. (3) **8**, 164–172 (1953).

In the differential equation

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right)+x^3f(y)=0, \quad \lambda>0,$$

let $f(y)$ satisfy the assumptions: (α) $f(0)=0$; (β) $0\leqq f'(y)\leqq L$ for $0\leqq y\leqq C$; (γ) $f(y)$ continuous and non-decreasing in $0\leqq y<\infty$. The author establishes the existence and uniqueness of a solution such that $\lim_{x\to+0}y(x)=C$ $(0<C<\infty)$, and $y(x)>0$, $y'(x)<0$, in $0<x\leqq x^*$. This is shown by reducing the differential equation to a non-linear integral equation of Volterra type and applying the Picard iteration technique to the latter. By comparison with the solutions in the special cases $f(y)=$const. and $f(y)=y^n$ the author then derives a lower bound for the zeros of this solution, as well as conditions sufficient for the existence or non-existence of such zeros. *W. Wasow.*

Popov, B. S. On Weber's differential equation. Proc. Indian Acad. Sci. Sect. A. **38**, 64–66 (1953).

In the differential equation $y''+Q_1y'+Q_2y=0$, let Q_1 and Q_2 be quadratic polynomials of the independent variable x. The author obtains a condition (involving n and the coefficients of Q_1, Q_2) which ensures that

$$D^2+Q_1D+Q_2=\left(D+f_1+\frac{f_2'}{f_2}\right)\left(D+\frac{f_2}{f_2}\right), \quad D=\frac{d}{dx},$$

where f_1 is a linear function of x, f_2 is a polynomial of degree n with leading coefficient 1, and f_3 is a polynomial of degree $n+1$. He uses this factorization to solve the differential equation, and applies this to Weber's equation

$$y''=\tfrac{1}{4}(x^2+4n+2)y$$

for which he obtains the factorization

$$\left(D+n\frac{H_{n-1}}{H_n}\right)\left(D+\frac{H_{n+1}}{H_n}\right)y=0.$$

A. Erdélyi (Pasadena, Calif.).

Pipes, Louis A. Matrix solution of equations of the Mathieu-Hill type. J. Appl. Phys. **24**, 902–910 (1953).

The author starts from Hill's second-order linear homogeneous differential equation without first-order term, the coefficient of the dependent variable being a periodic function of the independent variable. Considering the solution consisting of two linearly independent integrals, a matrix is formed from this solution and its first derivative. The value of this matrix at the end of the fundamental period is essentially the product of two matrices, starting from the initial conditions. A short summary of matrix operations and definitions is given. Then the method is applied to Meissner's equation and his results are reproduced. Next, the periodic function is taken to be the sum of a number of step functions and the problem is solved by the matrix method. A numerical example is given by way of illustration. Next, a Hill equation with exponential variation of the periodic coefficient is solved by the matrix method, using Bessel functions. Finally, Bessel functions of order one-third are applied in conjunction with the matrix method to the solution of a Hill equation with saw tooth coefficient. The closing paragraph deals with asymptotic approximations. *M. J. O. Strutt* (Zurich).

Matsumoto, Toshizô. Note on nonlinear differential equation of catalysis. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 267–270 (1953).

The author shows that the equation

$$\frac{d}{dx}\left(\frac{x^3}{y+1}\frac{dy}{dx}\right)=n^2x^2y$$

has an even-term power series expansion, and indicates a method for calculating its coefficients. From the expansion he produces a majorizing function which gives a lower bound for the abscissa of convergence of the expansion.

E. Pinney (Berkeley, Calif.).

Dahmen, Gert. Die elementare Auflösbarkeit der speziellen Riccatischen Differentialgleichung. Ann. Univ. Saraviensis **2**, 75–81 (1953).

For the necessary and sufficient condition that the original Riccati equation be solvable by elementary functions this paper gives a proof which is simpler than Liouville's. Ritt's simplification [C. R. Acad. Sci. Paris **183**, 331–332 (1926)] of one step is cited and incorporated. The author, however, seems unaware of Ritt's later complete treatment [Integration in finite terms, Columbia Univ. Press, 1948, pp. 70–76; these Rev. **9**, 573] which has much in common with the present proof and is at least as simple. *J. M. Thomas.*

Piaggio, H. T. H. Exceptional integrals of a not completely integrable total differential equation. Proc. Glasgow Math. Assoc. **1**, 137–138 (1953).

The usual theory of a single Pfaffian equation holds if the coefficients are of class C'. The author effectively remarks that this is not a necessary condition for the existence of a solution. As an example he gives the Pfaffian $z^m dx+z^n dy+dz$, $0<m$, $0<n$, $m+n=1$, $m\neq n$, which has integral surface $z=0$, although the coefficients are not differentiable with respect to z on that surface. Such a solution he calls exceptional to distinguish it from the singular integral varieties which appear in the usual theory and which make the coefficient of $\omega\omega'$ zero, ω being the Pfaffian (here of class three) and ω' its derived form. *J. M. Thomas.*

Birindelli, Carlo. Integrazione dei sistemi lineari ai differenziali totali illimitatamente integrabili in due variabili indipendenti in un prescritto campo più volte connesso. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 386–390 (1953).

Soit le système aux différentielles totales

$$dx_s=(g_s^{(1)}+x_k\phi_s^{1,k})dt_1+(g_s^{(2)}+x_k\phi_s^{2,k})dt_2,$$

$s=1, 2, \cdots, p$, où les g, ϕ sont $(3+2p)p$ fonctions données des deux variables indépendantes t_1, t_2, continues ainsi que leurs dérivées premières dans un champ T du plan, système que l'on suppose satisfaire dans ce champ aux conditions d'intégrabilité complète. L'auteur a étudié précédemment le cas où T est simplement connexe [mêmes Rend. (8) **12**, 518–523 (1952); ces Rev. **14**, 278]. Il arrive ici au résultat suivant: "La recherche des solutions pour un champ $\nu+1$ fois connexe est ramenée à la résolution d'un certain système de νp équations algébriques linéaires à p inconnues". Il laisse de côté ici la discussion des divers cas qui peuvent se présenter et se contente d'énoncer le théorème: Si le système admet une solution quels que soient les termes tout connus g, la solution est unique. *M. Janet* (Paris).

Redheffer, Raymond M. Basis theorems for partial differential equations. Duke Math. J. **20**, 489–498 (1953).

Consider a partial differential equation $f=0$, and a set of solutions which is a vector space over the field of real numbers. There exists a finite system of homogeneous linear partial differential equations such that every function in the vector space is a solution of the system, and every function which satisfies the system at a point satisfies $f=0$ at that point. After establishing this elementary fact, the author investigates conditions under which the coefficients in the system can be taken continuous. *E. R. Kolchin.*

Carrelli, A. Sul problema della separabilità delle variabili. Nuovo Cimento (9) **10**, 1213–1218 (1953).

Known results concerning separability are referred to [see also Moon and Spencer, these Rev. **14**, 173, 756] and given quantum mechanical interpretations. General results due to Agostinelli are considered and a geometrical analogy is given between a case treated by Levi-Cività and the classical one. *T. E. Hull* (Vancouver, B. C.).

Yurtsever, Berki. Lösung einer partiellen Differentialgleichung durch unendliche Reihen. Communications Fac. Sci. Univ. Ankara. Sér. A. **4**, 1–39 (1952). (Turkish summary)

The author obtains a solution of the equation:

$$\partial^\lambda z/\partial y^\lambda=\partial^\mu z/\partial x^\mu \quad (\lambda>\mu)$$

under the boundary conditions:

$$z_{x^k}(0, y)=k!f_k(y) \quad (k=0, 1, \cdots, \mu-1)$$

provided that certain series formed from the derivatives of the known functions f_k can be continued analytically along the positive real axis and satisfy certain summability conditions. The solution is expressed both as an infinite series and as an infinite integral, the latter involving the calculation of a considerable number of auxiliary quantities. He illustrates these calculations and the verification of the hypotheses by examples, when $\lambda=2$, $\mu=1$, when $\lambda=3$, $\mu=2$, and when $\lambda=3$, $\mu=1$. *D. Bernstein* (Rochester, N. Y.).

Hartman, Philip, and Wintner, Aurel. On elliptic Monge-Ampère equations. Amer. J. Math. **75**, 611–620 (1953).

If $f(x, y, z, p, q)$ is analytic in some 5-dimensional domain and $z(x, y)$ is a solution of class C^2 of the equation $rt-s^2=f(x, y, z, p, q)$, then $z(x, y)$ is analytic. The proof is based on the following result: If the partial derivatives of order n of $\phi(x, y)$ exist and satisfy a Lipschitz condition with an exponent λ, $0<\lambda<1$, then a solution $z(x, y)$ of $rt-s^2=\phi(x, y)$ possesses all $(n+2)$th partial derivatives and these satisfy Lipschitz conditions with any exponent $\mu<\lambda$.

H. Busemann (Los Angeles, Calif.).

Levitan, B. M. On expansion in characteristic functions of the Laplace operator. Doklady Akad. Nauk SSSR (N.S.) **90**, 133–135 (1953). (Russian)

Let $\mu_1^2, \mu_2^2, \cdots$ and $\omega_1, \omega_2, \cdots$ be the eigenvalues and orthonormalized eigenfunctions of a vibrating membrane covering a region D in m-space. Put

$$\theta(P, Q, \mu)=\sum_{\mu_n<\mu}\omega_n(P)\omega_n(Q),$$

$$\theta_s(P, Q, \mu)=\Gamma^{-1}(1+s)\sum_{\mu_n<\mu}\left(1-\frac{\mu_n^2}{\mu^2}\right)^s\omega_n(P)\omega_n(Q)$$

and

$$\theta_s^*(P, Q, \mu)=2^s(2\pi)^{-m/2}\mu^{\frac{1}{2}m-s}r^{-\frac{1}{2}m-s}I_{\frac{1}{2}m+s}(\mu r) \quad (r=|PQ|).$$

Let D_ϵ be D less a boundary strip of width $\epsilon>0$ and let P and Q be in D_ϵ. The following results are announced. There exists a constant C such that

$$\sum_{\alpha<\lambda_n<\alpha+1} |\omega_n(P)\omega_n(Q)| < C|\alpha|^{m-l},$$

$\theta_s-\theta_s^*=O(\mu^{m-l-s})$ or $O(\mu^{l-s})$ or $O(\mu^{-l})$ according as $l=[s]\leqq m-1$, $m-1<l\leqq m$, or $l>m$. If $s=0$, the first formula improves somewhat a result by Carleman [Åttonde Skandinaviska Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, p. 34]. Let $c_n=\int f\omega_n dP$ be the Fourier coefficients of a function f in $L^2(D)$ and consider the Riesz mean value

$$R_s(P,\mu)=\Gamma^{-1}(1+s)\sum_{\mu_n<\mu}\left(1-\frac{\mu_n^2}{\mu^2}\right)^s c_n\omega_n(P)$$

and the corresponding Riesz mean value $R_s^*(P,\mu)$ of the Fourier transform of f ($f=0$ outside D). Then if $s=\frac{1}{2}(m-1)$, $R_s-R_s^*$ tends to 0 uniformly in D_ϵ. Let $s>\frac{1}{2}(m-1)$ and put $l=[s]$. Then $R_s-R_s^*=O(\mu^{-l})$, $O(\mu^{s-l})$ or $O(\mu^{l-s})$ uniformly in D_ϵ according as $l\geqq\frac{1}{2}(m+1)$, $l\leqq\frac{1}{2}(m-1)$ or $\frac{1}{2}(m-1)<l\leqq\frac{1}{2}(m+1)$. From a result by Bochner [Trans. Amer. Math. Soc. 40, 175–207 (1936)] it then follows that $R_s\to f$ at the points of continuity of f. *L. Gårding.*

*Szegö, Gabor. **On the vibrations of a clamped plate.** Atti del Quarto Congresso dell'Unione Matematica Italina, Taormina, 1951, vol. II, pp. 573–577. Casa Editrice Perrella, Roma, 1953.

Rayleigh has conjectured that among all clamped plates with a given area, the circular plate has the least first eigenvalue. The conjecture was proved by Szegö under the assumption that the corresponding eigenfunction has no zeros. This is true if Green's function for the plate or any of its iterates is positive, but it is announced that recent simple examples by Loewner, Szegö and Garabedian show that there are domains for which this is not true.

L. Gårding (Lund).

Dikiĭ, L. A. On a formula of Gel'fand-Levitan. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 119–123 (1953). (Russian)

An alternative proof of a result of Gelfand and Levitan [Doklady Akad. Nauk SSSR (N.S.) **88**, 593–596 (1953); these Rev. **15**, 33]. *N. Levinson* (Cambridge, Mass.).

Hellwig, Günter. Anfangs- und Randwertprobleme bei partiellen Differentialgleichungen von wechselndem Typus auf den Rändern. Math. Z. **58**, 337–357 (1953).

The equation $B(u)+k(x)\{r_1u_{yy}+r_2u_y+r_3u\}=k(x)f(x,y)$, where $r_1>0$, r_2, r_3 are constants, $B(u)=-(p(x)u_x)_x+q(x)u$, is considered in a domain $l\leqq x\leqq m$, $0\leqq y<\infty$. For this equation the initial and boundary conditions discussed are: Cauchy data on the segment $l\leqq x\leqq m$ of the x-axis and boundary-values

$$\lim_{x\to l} u(x,y)\cos\delta+\lim_{x\to l} u_x(x,y)\sin\delta=0,$$

$$\lim_{x\to m} u(x,y)\cos\theta+\lim_{x\to m} u_x(x,y)\sin\theta=0.$$

If $p(x)$ and $k(x)$ are positive the equation is of hyperbolic type. The author proves existence theorems in the case where the equation is of hyperbolic type in the strip $l<x<m$, $0\leqq y<\infty$ and the quantity $k/p\to 0$ or ∞ as $x\to l$. In addition to the usual conditions on the smoothness of the coefficients and initial values certain assumptions as to the behavior of the differential operator $B(u)$ are made. These concern the behavior of $B(u)$ considered as an ordinary differential operator involving only the variable x. In this way the author is able to generalize the techniques used for ordinary equations. The results apply also to the case where $r_1=0$, $r_2>0$, i.e., when the equation is parabolic.

M. H. Protter (Berkeley, Calif.).

*Cinquini-Cibrario, Maria. **Un teorema fondamentale per la teoria delle caratteristiche di equazioni non lineari di ordine n di tipo iperbolico.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 52–56. Casa Editrice Perrella, Roma, 1953.

Suppose that the equation $F(x,y;z;p_{rs})=0$, with $r+s=1,\cdots,n$ and $p_{rs}=\partial^{r+s}z/\partial x^r\partial y^s$, is hyperbolic, in an appropriate sense, in a region D. In the analytic case it was shown by Goursat that there exists a unique solution surface containing both a given characteristic strip and a given curve cutting (non-tangentially) the curve of the strip. A very similar theorem, based on earlier work of the author, is announced here for the real case. *F. A. Ficken.*

Garnir, H. G. Fonctions de Green de l'opérateur métaharmonique pour les problèmes de Dirichlet et de Neumann posés dans un angle ou un dièdre. Bull. Soc. Roy. Sci. Liège **21**, 119–140, 207–231 (1952).

The Green's function and the Neumann's function for the operator Δ^2-k^2, either in the plane region bounded by the lines $\theta=0$, $\theta=\alpha$ or in the three-dimensional region bounded by the planes $\theta=0$, $\theta=\alpha$, is obtained by an image method from the periodic Green's function for these regions. The periodic Green's function is a solution of $(\Delta^2-k^2)G=0$ which has the appropriate point-source singularity and which satisfies the following boundary condition: The values of G and its normal derivative for $\theta=0$ equal the corresponding values for $\theta=\alpha$. This periodic Green's function is obtained by separation of variables using a transform in the radial direction. The appropriate inversion formula for this transform was given by Kontorowitch and Lebedev [Acad. Sci. USSR. J. Phys. **1**, 229–241 (1939)]. The result is then expressed as a Fourier transform with respect to the separation variable. The representations obtained by the author are similar to those derived by Carslaw [Philos. Mag. (6) **5**, 374–379 (1903)] using Sommerfeld's theory of images in Riemann spaces [Proc. London Math. Soc. (1) **28**, 395–429 (1897)]. Finally, the author makes a detailed examination of the results in the cases where $\alpha=2\pi/n$ or $4\pi/n$, $n=1,2,3\cdots$. *B. Friedman.*

Garnir, H. G. Fonctions de Green des opérateurs

$$\Delta-k^2,\quad (k>0),\quad \Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2},\quad (c>0),\quad \Delta-\frac{1}{k}\frac{\partial}{\partial t},\quad (k>0),$$

pour les problèmes de Dirichlet et de Neumann posés dans un segment, une bande ou une dalle. Bull. Soc. Roy. Sci. Liège **22**, 29–46 (1953).

The author obtains several representations of the Green's functions for the operators mentioned in the title. The regions considered are the line, strip, or slab defined by $0<K<d$, in one, two, or three dimensions, respectively. The results are well-known and may be found by standard techniques such as Laplace transforms and separation of variables. *B. Friedman* (New York, N. Y.).

Garnir, H. G. Propagation de l'onde émise par une source ponctuelle et instantanée dans un dioptre plan. Bull. Soc. Roy. Sci. Liège **22**, 85–100, 148–162 (1953).

The mathematical problem is this: Find $G(P, P_0, t)$ such that

$$(*) \qquad \Delta^2 G - \frac{1}{C^2}\frac{\partial G}{\partial t^2} = \delta(P, P_0)\delta(t),$$

where the Laplacian is to be taken in 1, 2, or 3 dimensions, where C^2 is a constant C_+^2 for $X>0$ and another constant C_-^2 for $X<0$, and where $\delta(P, P_0)$ is the Dirac delta-function of P with source point P_0. The boundary and initial conditions on G are

$$G(P, P_0, 0) = G_t(P, P_0, 0) = 0,$$
$$\beta_+ G(X=0+) = \beta_- G(X=0-)$$

and $\alpha_+ G_x(X=0+) = \alpha_- G_x(X=0-)$ where β_+, β_-, α_+, α_- are given constants. The solution is obtained by taking the Laplace transform of (*) with respect to t, thus reducing it to $\Delta^2 g - k^2 g = \delta(P, P_0)$. This is solved by standard methods using separation of variables and then by inverting the Laplace transform the solution of (*) is obtained. The result is written as the sum of three terms: the direct effect of the source, the effect of the reflection from $X=0$ and the effect transmitted across $X=0$. Finally, the author uses some intricate contour integrations to show that these solutions give the results which could be obtained by geometrical optics. *B. Friedman* (New York, N. Y.).

Müller, Claus. Über die ganzen Lösungen der Wellengleichung. Math. Ann. **124**, 235–264 (1952).

Let a) P be a point in the p-dimensional Euclidean space defined by a vector $\mathfrak{x}=(x_1, \cdots, x_p)$; $r=(x_1^2+\cdots+x_p^2)^{1/2}$ the length of $\mathfrak{x}$ and $\mathfrak{x}_0=r^{-1}\mathfrak{x}$ a point on the unit-hypersphere Ω with the surface element $d\omega$, b) $U(\mathfrak{x})$ be an entire solution of the wave equation $\Delta U+U=0$ in p dimensions such that

$$r^{p-1}\int_\Omega |U(\mathfrak{x})|^2 d\omega \leqq M^2;$$

then an asymptotic expansion for large r is given for $r^{(p-1)/2}U_h(\mathfrak{x})$, where $U_h(\mathfrak{x})$ denotes an expression which is obtained by subjecting U h times to a well-defined integral operation over a region of the unit sphere Ω. For $h=2$ this theorem contains a result previously established by Herglotz. The general method applied here admits furthermore a proof of the theorem: Each entire solution of $\Delta U+U=0$, provided that $r^{p-1}\int_\Omega |U(\mathfrak{x})|^2 d\omega \leqq M^2$ and $\lim_{r\to\infty} r^{p-1}\int_{\Omega_+} |U(\mathfrak{x})|^2 d\omega = 0$, vanishes identically. This is Magnus' theorem [Abh. Math. Sem. Univ. Hamburg **16**, nos. 1–2, 77–94 (1949); these Rev. **11**, 176, 871]. *F. Oberhettinger* (Washington, D. C.).

Nitsche, Johannes, und Nitsche, Joachim. Bemerkungen zum zweiten Randwertproblem der Differentialgleichung $\Delta\varphi=\varphi_x^2+\varphi_y^2$. Math. Ann. **126**, 69–74 (1953).

Let T be a bounded simply connected plane region having a boundary curve S with three continuous derivatives. Let n denote the inner normal to S, and let $f(s)$ have a continuous derivative $\rho(s)$ on S. If $\int_S \rho(s)ds=0$, it is possible to determine a constant M such that if $\max|\rho(s)| \leqq M$ then there exists a unique value of the constant c such that the problem R_c: $\varphi_{xx}+\varphi_{yy}=\varphi_x^2+\varphi_y^2$ in T, $\varphi_n=-f(s)-c$ on S, has a unique solution φ with φ_x and φ_y continuous in $T+S$ and φ_{xx} and φ_{yy} continuous in T. For any other value of c, R_c is inconsistent. Thus R_c is "inappropriate" for all but one value of c. It is pointed out that consistency conditions in linear problems can ordinarily be given a priori, while in this nonlinear problem the consistency condition for R_c is $c=l^{-1}[\iint_T \varphi_x^2+\varphi_y^2)dxdy-\int_S f(s)ds]$ where l is the length of S, and thus involves a knowledge of the "solution" φ. The method is to transform R_c first to an elliptic first-order system and then to a pair of nonlinear integral equations (in which c does not appear) whose iterative solution, using estimates due to Korn, requires $\max|\rho(s)| \leqq M$.

F. A. Ficken (Knoxville, Tenn.).

Pini, Bruno. Sul primo problema di valori al contorno della teoria dell'elasticità. Rend. Sem. Mat. Univ. Padova **21**, 345–369 (1952).

G. Cimmino [Rend. Circ. Mat. Palermo **61**, 177–221 (1938); Rend. Sem. Mat. Univ. Padova **11**, 28–89 (1940); Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 322–329 (1940); these Rev. **8**, 270; **1**, 314] has formulated and proved existence and uniqueness theorems for a generalized Dirichlet problem for second order linear elliptic partial differential equations. The present author formulates a similar "Dirichlet type" problem for the equations of elasticity, and proves the corresponding existence and uniqueness theorems, following Cimmino's methods. The boundary value problem consists in the determination of the displacement vector u, satisfying

$$\Delta u + k \operatorname{grad} \operatorname{div} u = 0, \quad \text{on} \quad D-S,$$

$$\lim_{t\to 0}\int\int_{S_t} |u-\bar{u}|^2 d\sigma = 0,$$

where D is a bounded domain, having as boundary a regular surface S, with continuous normal and continuous principal curvatures, admitting the parametric representation $x_i=\bar{x}_i(\alpha, \beta)$, $i=1, 2, 3$; the S_t is a family of parallel surfaces to S: $x_i=\bar{x}_i(\alpha, \beta)+t\nu_i(\alpha, \beta)$, $i=1, 2, 3$, $0 \leqq t \leqq t_0$, with $n=(\nu_1, \nu_2, \nu_3)$ the unit inner normal to S; and $\bar{u}$ is a given vector, of integrable square norm on the surface S.

J. B. Diaz (College Park, Md.).

Krzyżański, M. Sur le second problème aux limites pour les équations linéaires aux derivées partielles du type elliptique et parabolique dans un domaine non borné. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **5** (1951), 1–21 (1953). (Polish and Russian summaries)

Let D be a closed and unbounded domain with boundary FD, and let $\{D_n\}$ be a sequence of closed and bounded domains contained in D such that $D_i \subset D_{i+1}$ and $D=\lim_{n\to\infty} D_n$. Let $\Phi(P)\equiv\Phi(x_1, \cdots, x_n)$ be a function defined and continuous in D. Assume that the Dirichlet problem: u_n a solution of the elliptic equation

$$(1) \qquad E(u) \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + cu = f$$

interior to D_n and $u_n=\Phi(P)$ on the boundary of D_n is solvable for every D_n. Under certain additional assumptions the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 408–416 (1948); these Rev. **10**, 254] has shown that $u=\lim_{n\to\infty} u_n$ leads to a solution of the Dirichlet problem for domain D with $u=\Phi(P)$ on FD. The major portion of the present paper is devoted to proving that this type of approach, under similar restrictions, successfully handles the following boundary-value problem. Find in D a solution u of (1) which on the boundary FD satisfies the condition $\alpha(P)du/dl+\beta(P)u=g(P)$. Here du/dl represents a directional derivative taken along a line segment internal to D at the point P of FD. An analogous problem is handled for the parabolic equation $E(u)-\partial u/\partial y=f$.

F. G. Dressel (Durham, N. C.).

Fukuda, Nobuji, and Yosikawa, Teruya. Conduction of heat in a cylinder composed of three different materials. J. Osaka Inst. Sci. Tech. Part I. 2, 53–65 (1950).

Let $\theta(r, t)$ denote the temperatures in a composite cylinder of infinite length formed by inclosing a cylinder of one material by two coaxial cylindrical shells of other materials. Each material is uniform. Perfect thermal contact is assumed at the two cylindrical interfaces. Under the initial condition $\theta(r, 0)=0$ and the condition $\theta(c, t)=\theta_0$ where $r=c$ is the outer boundary and θ_0 is a constant, the lengthy formulas for $\theta(r, t)$ in the three regions are computed with the aid of contour integration and residues. The characteristic equation whose roots are the eigenvalues that occur in those formulas is a very involved one. *R. V. Churchill.*

Egerváry, Jenő, et Lovass-Nagy, Viktor. La solution de l'équation différentielle de la conduction calorique avec condition périphérique dépendant linéairement de la durée. (Examen du processus de refroidissement respectivement de rechauffement d'un corps placé dans un médium dont la température varie d'une manière uniforme.) Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 1 (1952), 11–22 (1953). (Hungarian. Russian and French summaries)

The boundary-value problem solved in this paper is: $u_t=a^2\Delta u$, $u=0$ when $t=0$, $b\partial u/\partial n+u=v$ on the boundary (b, v constants), and describes conduction of heat in a body which is being heated. The solution is written out explicitly for a rectangular finite slab, and some graphs and numerical values are also given. *A. Erdélyi* (Pasadena, Calif.).

Agamirzyan, L. A. On a general representation of the solution of an equation connected with an axially symmetric problem of the theory of elasticity. Soobščeniya Akad. Nauk Gruzin. SSR 13, 385–388 (1952). (Russian)

The author considers the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=0,$$

which arises in axially symmetric elasticity. Setting $z=x+ir$ and $\zeta=x-ir$, this equation may be rewritten in complex form:

$$(*)\quad \left[\frac{\partial^2}{\partial z\partial\zeta}-\frac{1}{2}\frac{1}{z-\zeta}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\zeta}\right)\right]\left[\frac{\partial^2 u}{\partial z\partial\zeta}-\frac{1}{2}\frac{1}{z-\zeta}\left(\frac{\partial u}{\partial z}-\frac{\partial u}{\partial\zeta}\right)\right]=0.$$

I. N. Vekua [New methods of solution for elliptic equations, OGIZ, Moscow-Leningrad, 1948, p. 187; these Rev. 11, 598] has introduced the notion of the Riemann function for an equation

$$\sum_{k=0}^{n}\sum_{m=0}^{n}A_{km}(z,\zeta)\frac{\partial^{k+m}u}{\partial z^k\partial\zeta^m}=0.$$

Starting from the known Riemann function for the Euler-Poisson equation

$$\frac{\partial^2\varphi}{\partial z\partial\zeta}-\frac{1}{2}\frac{1}{z-\zeta}\left(\frac{\partial\varphi}{\partial z}-\frac{\partial\varphi}{\partial\zeta}\right)=0$$

the author determines the Riemann function, in Vekua's sense, for the partial differential equation (*).

J. B. Diaz (College Park, Md.).

Functional Analysis, Ergodic Theory

***Riesz, Frédéric, et Sz.-Nagy, Béla. Leçons d'analyse fonctionnelle.** 2ème éd. Akadémiai Kiadó, Budapest, 1953. viii+455 pp. 100 ft.

This edition differs from the first only in correction of known misprints and errors, rewriting of parts of certain sections (e.g., §§17, 22, 138, 139, 142, 152), an added new section (§143) in chapter X, and additions to the bibliography [cf. these Rev. 14, 286].

Stone, M. H. On the theorem of Gelfand-Mazur. Ann. Soc. Polon. Math. 25 (1952), 238–240 (1953).

Let A be a convex topological linear division algebra over the complex field C, with continuous inversion. The Gelfand-Mazur theorem, as extended by the reviewer, states that A is isomorphic to C. The author shows that, for the case in which A is complete, a proof can be given which avoids direct appeal to the analytic function theory, in particular to Liouville's theorem. The proof is based on the discovery of some impossible behaviour of $a^n(a^n-r^n)^{-1}$ as n tends to infinity, $r\geqq 0$, if a is an element which is supposed not to be a complex multiple of the unit. *R. Arens.*

Cooper, J. L. B. Coordinated linear spaces. Proc. London Math. Soc. (3) 3, 305–327 (1953).

L'auteur étudie dans ce travail une catégorie d'espaces vectoriels topologiques qui comprend comme cas particuliers les espaces de suites étudiés sous le nom de "vollkommene Raüme" par G. Köthe, ainsi que la généralisation de ces espaces considérés par le rapporteur sous le nom d'"espaces de Köthe" [J. Analyse Math. 1, 81–115 (1951); ces Rev. 12, 834]. La définition donnée par l'auteur se rattache à sa théorie des "mailles" (meshes) développée par lui antérieurement [Proc. London Math. Soc. (2) 52, 220–240 (1951); ces Rev. 12, 728] et ne saurait être reproduite ici. Mais cette définition équivaut à la suivante: on considère un espace mesuré T, et pour chaque $t\in T$, un couple d'espaces localement convexes séparés $(E(t), E'(t))$ en dualité; l'auteur considère alors un espace E formé de "champs de vecteurs" x, qui, à chaque $t\in T$, associent un élément $x(t)\in E(t)$, et à un tel espace associe l'espace E' des champs de vecteurs x' tels que, pour chaque $t\in T$, $x'(t)\in E'(t)$, de sorte que la fonction $\langle x(t), x'(t)\rangle$ soit intégrable dans T (en réalité, il semble que certaines restrictions supplémentaires soient apportées à la mesure sur T). L'auteur aborde alors pour ces espaces les questions traitées par Köthe dans le cadre des "vollkommenne Raüme": enveloppe normale des ensembles bornés ou faiblement compacts dans E, caractérisation de la topologie de Mackey $\tau(E, E')$ sur E, propriétés de "complétion" pour les diverses topologies sur E. Le mémoire se termine par un critère de continuité pour des applications linéaires d'un espace $L^1(\mu)$ dans un espace localement convexe. *J. Dieudonné.*

Köthe, Gottfried. Dualität in der Funktionentheorie. J. Reine Angew. Math. 191, 30–49 (1953).

The general aim of this paper is to develop the ideas and methods of linear topological spaces for certain particular linear spaces whose elements are analytic functions. Let Ω be the extended complex plane, G an open connected subset of Ω, and $P(G)$ the class of all functions analytic on G, with $f(\infty)=0$ if $\infty\in G$. If $x\in P(G)$ and H is a subset of G closed in Ω, let $\|x\|_H=\sup_{z\in H}|x(z)|$. The sets $\|x\|_H<\epsilon$ with variable ϵ and H, are taken as a fundamental system of neighborhoods of the origin in $P(G)$, and this makes $P(G)$ an (F)

space. It is also an (M) space in the terminology of Dieudonné and Schwartz [Ann. Inst. Fourier Grenoble 1, 61–101 (1950); these Rev. 12, 417]; that is, every bounded closed set in $P(G)$ is compact. Next let O be any open proper subset of Ω with components $G^{(1)}, G^{(2)}, \cdots$. Define $P(O)$ as the class of functions x defined on O such that the restriction of x to $G^{(i)}$ is an element of $P(G^{(i)})$. Give $P(O)$ the topology of the topological product of the $P(G^{(i)})$; then $P(O)$ is also an (F) and (M) space. If A is a closed set in Ω, consider open sets which contain A and each of whose components intersects A. Two analytic functions, each one defined on such an open set, are called equivalent if they coincide on the intersection of the sets. Denote by $R(A)$ the space whose elements are classes of equivalent functions. This space is given a topology by considering it as the union of a certain countable set of Banach spaces $B_1, B_2, \cdots$. If W_n is a neighborhood of the origin in B_n and k is any positive integer, the elements $\alpha_1 u_1 + \cdots + \alpha_k u_k$ with $u_n \in W_n$ and $\sum_1^k |\alpha_n| \leq 1$ form a neighborhood of the origin in $R(A)$. It is shown that each of the spaces $P(O)$, $R(\Omega - O)$ can be identified as the dual of the other, the dual of a space being the set of all continuous linear functionals defined on it. As remarked by the author in a footnote appended at the proofreading stage, this duality relationship was proved independently by C. da Silva Dias [Thesis, Univ. of São Paulo, 1951; these Rev. 13, 249], and also by A. Grothendieck. A number of further relations between $P(O)$ and $R(\Omega - O)$ follow from the paper of Dieudonné and Schwartz already referred to. These spaces are weak duals of each other. Their strong topologies coincide with the topologies as originally defined. Weak and strong convergence of sequences are the same things in each space. A linear subspace of $R(\Omega - O)$ is weakly closed if it is sequentially closed.

The latter part of the paper deals with linear mappings of $P(O_1)$ into $P(O_2)$. Each such mapping determines and is determined by a function $a(\lambda_1, z_2)$ analytic on $A_1 \times O_2$, where $A_1 = \Omega - O_1$. The mapping $x_1 \to x_2$ is then given by $x_2(z_2) = (1/2\pi i) \int x_1(t_1) a(t_1, z_2) dt_1$, with the integral taken over a suitable path. Corresponding results hold for linear mappings of $R(A_2)$ into $R(A_1)$. Relations between weak and strong continuity are considered for such mappings. Finally, mappings defined by differentiation, integration, or multiplication by a function are considered as examples.

A. E. Taylor (Los Angeles, Calif.).

***Frehner, Hedi, e Pellegrino, Franco. Sulla topologia delle regioni funzionali lineari.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 105–107. Casa Editrice Perrella, Roma, 1953.

Dans cette note on considère deux topologies définies dans l'ensemble obtenu à partir d'une région linéaire (C) lorsqu'on identifie chaque fonction $f \in (C)$ avec tous ses prolongements analytiques. Les auteurs oublient que cette identification a été signalée en 1946 comme point de départ d'une nouvelle systématisation de la théorie des fonctionnelles analytiques [Sebastião e Silva, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. (8) 1, 709–715 (1946); Portugaliae Math. 9, 1–130 (1950); ces Rev. 8, 278; 11, 524; voir aussi L. Nachbin, Revista Unión Mat. Argentina 12, 129–150 (1947); ces Rev. 9, 367]. Une mise au point de cette systématisation, à la lumière de la théorie des espaces vectoriels topologiques, a été faite en 1951 indépendamment par G. Köthe [voir l'analyse ci-dessus], C. da Silva Dias [Thèse, Univ. de São Paulo, 1951; ces Rev. 13, 249] et A. Grothendieck [dans un mémoire à paraître]. D'après les résultats de Köthe on peut affirmer que l'espace quotient de la région (C), munie de la topologie de Fantappiè, par la relation d'équivalence considérée, est un espace localement convexe; en particulier, il est un groupe topologique. Les auteurs signalent ce dernier fait, dont il serait intéressant de connaître une démonstration directe. *J. Sebastião e Silva* (Lisbonne).

***Del Pasqua, Dario, e Pellegrino, Franco. Principi di una teoria degli operatori lineari.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 77–81. Casa Editrice Perrella, Roma, 1953.

Pour chaque opérateur linéaire L admettant un inverse à gauche $\bar{L}$, les auteurs considèrent l'opérateur $Z = I - L\bar{L}$ (I étant l'identité) et ils indiquent, sans démonstration, quelques résultats dans cet ordre d'idées, en se rapportant à l'espace $\mathfrak{S}$ de Fantappiè. [Note du reviewer: les régions linéaires de $\mathfrak{S}$ ne sont pas des espaces vectoriels, ce qui donne lieu à certaines difficultés dont les auteurs semblent ne pas se rendre compte; par exemple, ils utilisent un zéro inexistant.] *J. Sebastião e Silva* (Lisbonne).

***Evans, Elisabetta, e Pellegrino, Franco. Sulla geometria delle regioni funzionali lineari.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 96–98. Casa Editrice Perrella, Roma, 1953.

Fantappiè a démontré que chaque région linéaire $\mathfrak{H}$ de son espace est déterminée par un sous-ensemble fermé C de la sphère de Riemann, Ω, les éléments de $\mathfrak{H}$ étant les fonctions $f \in \mathfrak{S}$ dont les domaines contiennent C. On voit alors que le réticulé des sous-ensembles fermés de Ω détermine un réticulé isomorphe de sous-espaces de $\mathfrak{S}$. Les auteurs signalent ce fait sans employer le langage de la théorie des réticulés. *J. Sebastião e Silva* (Lisbonne).

***Pellegrino, Franco, e Rugini, Franco. Sulla uniforme e semiuniforme continuità dei funzionali analitici lineari.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 169–172. Casa Editrice Perrella, Roma, 1953.

Dans cette note les auteurs définissent des concepts de "continuité uniforme" et de "continuité semi-uniforme" et ils indiquent quelques résultats qui en découlent. Dans une région linéaire A le premier de ces concepts se ramène à celui de continuité par rapport à la semi-norme $p(f) = \sup_{z \in A} |f(z)|$ (pour les fonctionnelles linéaires). *J. Sebastião e Silva.*

***Pellegrino, Franco, e Varsano, Sami. Sulle regioni di definizione dei funzionali analitici non lineari.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 176–180. Casa Editrice Perrella, Roma, 1953.

Il s'agit ici de quelques considérations générales sur les domaines de définition et sur le prolongement des fonctionnelles analytiques. *J. Sebastião e Silva* (Lisbonne).

***Pellegrino, Franco, e Succi, Francesco. Su alcune proprietà dei funzionali analitici misti e degli operatori da essi determinati.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 173–175. Casa Editrice Perrella, Roma, 1953.

On nomme ici "opérateurs" les transformations du type $f \to g$, où f, g sont des fonctions appartenant à l'espace fonctionnel analytique $\mathfrak{S}$; et "fonctionnelles mixtes" les transformations du type $(f, t) \to z$, où t, z sont des nombres et $f \in \mathfrak{S}$. Dans la théorie de Fantappiè les opérateurs se présentent déguisés sous la forme de fonctionnelles mixtes.

La présente note résume les premiers essais d'une étude de ces deux types de transformations au point de vue topologique (domaines d'existence, continuité, etc.).

J. Sebastião e Silva (Lisbonne).

Zeller, Karl. *FK*-Räume in der Funktionentheorie. II. Math. Z. **58**, 414–435 (1953). See the concluding note on p. 439.

The author contends that the theory of *FK* spaces rather than that of Banach spaces is the natural source of proving the existence of analytic functions with prescribed singularities. This is justified by the fact that the space $\mathfrak{R}$ of functions $\mathfrak{a}(z)=\sum a_k z^k$ analytic in $|z|<1$, its subspace of functions bounded on a ray $\arg z=\phi$, and other similar spaces are *FK* but not Banach spaces. *FK* spaces are spaces of complex sequences $\mathfrak{a}=\{a_k\}$ with a metric defined by a sequence of pseudo-norms, in which the mappings $\mathfrak{a}\to a_k$ are continuous. The methods used were outlined in the first part of this paper [Zeller, Math. Z. **58**, 288–305 (1953); these Rev. **14**, 1092]: (a) a theorem on comparison of pseudo-norms on two *FK* spaces $\mathfrak{A}$, $\mathfrak{B}$ which allows one to decide when $\mathfrak{A}\subset\mathfrak{B}$; (b) the fact that certain functions ("minorants of a pseudo-norm") are either bounded on an open set or infinite on a fat set. Following topics are treated. (1) Connection of unicity theorems with convergence theorems for functions in a subspace $\mathfrak{E}$ of $\mathfrak{R}$ [here the author's methods seem not to apply to H_p, $p<1$], weak convergence. (2) The existence of a function $\mathfrak{a}(z)\in\mathfrak{R}$ tending uniformly to ∞ along each radius [Lusin and Privaloff, Ann. Sci. Ecole Norm. Sup. (3) **42**, 143–191 (1925)]. (3) Any denumerable set and any set F_σ on $|z|=1$ is the exact set of existence of the radial limit $\lim_{\rho\to 1-}\mathfrak{a}(\rho e^{i\phi})$ for some $\mathfrak{a}\in\mathfrak{R}$; the radial limits may exist everywhere, but not uniformly on any arc [for convergence on $|z|=1$ of $\sum a_n z^n$ instead of radial limits this was given by Herzog and Piranian, Duke Math. J. **16**, 529–534 (1949); **20**, 41–54 (1953); these Rev. **11**, 91; **14**, 738]. (c) By means of Fejér's example it is shown that the estimate $\sum_{k=0}^{n}a_k=o(\log n)$ is the best possible for $\mathfrak{a}(z)$ continuous in $|z|\leqq 1$; the estimate $\sum_{k=0}^{n}|a_k|=o(n^{1/2})$ is the best possible for functions $\mathfrak{a}(z)$ with uniformly convergent power series in $|z|\leqq 1$.

G. G. Lorentz (Detroit, Mich.).

Sargent, W. L. C. On some theorems of Hahn, Banach and Steinhaus. J. London Math. Soc. **28**, 438–451 (1953).

Let E be a normed linear space. A subset A of E is an α-set in E if there is a sequence of nowhere dense sets A_n such that $A=\bigcup A_n$, and that $x, y\in A_n$ implies $x\pm y\in A_{n+1}$. The set A is a β-set in E if A is not an α-set. Well-known theorems of Hahn, Banach and Steinhaus on sequences of bounded linear operators $T_n(x)$, $x\in E$, use the assumption that the space E is of second category in itself; they may be strengthened by making the weaker assumption that E is a β-set in itself. The author exhibits spaces of functions (integrable in the Denjoy or the Cauchy-Lebesgue sense) which are β-sets although of first category. Applying his convergence theorem for operators, he obtains the following result. In order that $x(t)K(s,t)$ be Denjoy integrable for $s\geqq 0$ and

$$\lim_{s\to\infty}\int_0^\infty K(s,t)x(t)dt=\int_0^\infty x(t)dt$$

for each Denjoy integrable $x(t)$, it is necessary and sufficient that $K(s,t)$ should have a finite essential variation V_s for $s\geqq 0$, that V_s be bounded for large s and that $\int_0^\lambda K(s,t)dt\to\lambda$ for $s\to\infty$.

G. G. Lorentz.

Nevanlinna, Rolf. Bemerkung zur Funktionalanalysis. Math. Scand. **1**, 104–112 (1953).

By deriving and applying the differential mean value theorem for the Fréchet differential, the author proves Banach space generalizations of the implicit function theorem and the openness of mappings. The central theorems are the following. 1) Let $y=y(x)$ be a single-valued mapping between Banach spaces and assume that

$$\text{(A)}\qquad \inf\{|\delta y(0,h)|\,|\,|h|=1\}=m>0$$

and

$$\text{(B)}\qquad \sup\{|\delta y(x,h)-\delta y(x,0)|\,|\,|x|\leqq\rho\leqq R\}<m,$$

where $\delta y(x,h)$ is the Fréchet differential. Then $y=y(x)$ is one-one in $|x|\leqq\rho$. 2) If (A) holds and if

$$\text{(B')}\qquad \sup\{|\delta y(x,h)-\delta y(x,0)|\,|\,|x|\leqq R\}\leqq M|x|$$

and (C) $\delta y(0,h)$ maps the domain completely onto the range space, then $|x|\leqq\frac{1}{2}\rho=\frac{1}{2}\min(mM^{-1},R)$ is mapped onto a set which covers $|y-y(0)|<\frac{1}{4}m\rho$. Applications to finite-dimensional cases are given.

B. R. Gelbaum.

Ruston, A. F. Formulae of Fredholm type for compact linear operators on a general Banach space. Proc. London Math. Soc. (3) **3**, 368–377 (1953).

The paper is concerned with setting up expressions of Fredholm determinant type associated with transformations of the form $Tx=x-\lambda Kx$, where K is a compact (completely continuous) linear transformation on a linear normed space $\mathfrak{B}$ to $\mathfrak{B}$. Previous procedures are due to A. D. Michal and R. S. Martin [J. Math. Pures Appl. (9) **13**, 69–91 (1934)] and the author [Proc. London Math. Soc. (3) **1**, 327–384 (1951); these Rev. **13**, 468]. The papers of F. Smithies [Duke Math. J. **8**, 107–130 (1941); these Rev. **3**, 47] and A. C. Zaanen [Compositio Math. **10**, 56–94 (1952); these Rev. **14**, 767] are suggestive. The Riesz theory of such a transformation asserts the existence of an index $m(\lambda)$, the maximum (finite) dimension of the subspace of $\mathfrak{B}$ of all x for which $(I-\lambda K)^n x=0$ for some n. Also the points for which $m(\lambda)\neq 0$ (the characteristic values of K) have ∞ as only limiting point. This provides the well-known suggestion of setting up the entire function $\Delta_0(\lambda)=\sum_n\Delta_0^n\lambda^n$, having the characteristic values as zeros of order $m(\lambda)$, as a generalization of the Fredholm determinant. The coefficients of the corresponding series $\Delta_k(\lambda)=\sum_n\Delta_k^n\lambda^n$ for the kth order minors are then determined by the formula $\Delta_k^n=\sum_{i=0}^{k}\Delta_0^{n-i}K_k^{i+1}$ where K_k^i were defined by an iterative process in terms of the transformation K in the author's paper cited above. In particular, for $k=1$, $K_1^i=K^{(i)}$ [cf. Lalesco, Théorie des équations intégrales, Hermann, Paris, 1912, p. 25]. It is then proved that the series for $\Delta_k(\lambda)$ are permanently absolutely convergent power series of transformations and perform the duties of the Fredholm determinants in the inversion of the functional equation $(I-\lambda K)x=y$. This mode of defining Fredholm determinants has the admitted disadvantage that it presupposes knowledge of the characteristic values of K.

T. H. Hildebrandt (Ann Arbor, Mich.).

Atkinson, F. V. On relatively regular operators. Acta Sci. Math. Szeged **15**, 38–56 (1953).

$\mathfrak{R}$ is a linear normed complete space, $\mathfrak{R}_1$ is the class of bounded operators on $\mathfrak{R}$ to $\mathfrak{R}$. Then T of $\mathfrak{R}_1$ is relatively regular if there exists an X of $\mathfrak{R}_1$ such that $TXT=T$. Then there exists also X' (e.g., XTX) such that $TX'T=T$ and $X'TX'=X'$ [cf. general reciprocal of E. H. Moore, General analysis, v. I, pt. I, Philadelphia, Pa., 1935, p. 203]. Ex-

amples of regular operators are: finite-dimensional T, T having a right (or left) inverse, T expressible in the form $S+V$, where S has an inverse and V is completely continuous (Riesz theory), T a projection operator. If T is regular, then $T\mathfrak{R}$ is closed and $Tf=g$ is soluble if and only if g is orthogonal to all solutions f^* of the adjoint homogeneous equation $f^*T=0$ in $\mathfrak{R}^*$ (adjoint of $\mathfrak{R}$). The paper is concerned mainly with the dimensions $\alpha(T)$ and $\beta(T)$ of the linear spaces of solutions of $Tf=0$ $f^*T=0$, where T is relatively regular and either $\alpha(T)$ or $\beta(T)<\infty$, generalizing the Riesz case where $\alpha(T)=\beta(T)=$ finite, and results for the case when $\alpha(T)\neq\beta(T)$, both finite, obtained by Z. I. Halilov [Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 163–176 (1949); these Rev. 11, 39], I. C. Gohberg [Uspehi Matem. Nauk. (N.S.) 7, no. 2(48), 149–156 (1952); these Rev. 14, 54], and the author [Mat. Sbornik N.S. 28(70), 3–14 (1951); these Rev. 13, 46]. There is the algebraic aspect, involving relations between the α and β for S, T and ST. Then for the perturbation problem there are theorems of the type: (a) if either $\alpha(T)$ or $\beta(T)$ is finite, there exists a $\rho>0$ such that if $\|T-T'\|<\rho$ then $\alpha(T')\leqq\alpha(T)$ and $\beta(T')\leqq\beta(T)$; (b) if $\alpha(T)$ ($\beta(T)$) is finite and V is completely continuous, then $\alpha(T+V)$, ($\beta(T+V)$) is finite; (c) if $\alpha(T)<\infty$ and A_λ is analytic in λ in the vicinity of $\lambda=0$ with values in $\mathfrak{R}_1$, and $A_0=0$, then there exists a $\rho>0$ and an integer $m'\leqq\alpha(T)$ such that $0<|\lambda|<\rho$ implies that $T-A_\lambda$ is relatively regular and $\alpha(T-A_\lambda)=m'$. If in addition $\beta(T)=\infty$, then $\beta(T-A_\lambda)=\infty$. From this one deduces that there exists an open connected region containing $\lambda=0$ throughout which $T-A_\lambda$ is relatively regular,

$$\alpha(T-A_\lambda)=A(T)<\infty \quad \text{and} \quad \beta(T-A_\lambda)=\beta(T)=\infty,$$

excepting at most for a sequence of isolated points. For a T_λ an integral function of λ with values in $\mathfrak{R}_1$, for any pair m, n there exists at most a sequence of open connected regions $\mathfrak{S}_r$ with the property that T_λ in each $\mathfrak{S}_r$ is relatively regular, $\alpha(T_\lambda)=m$ and $\beta(T_\lambda)=n$, excepting at most at a denumerable set of points. *T. H. Hildebrandt.*

Block, H. D. Construction of solutions and propagation of errors in nonlinear problems. Proc. Amer Math. Soc. 4, 715–722 (1953).

In the principal theorem the author concludes that under appropriate completeness and Lipschitz conditions, a function F maps one metric abelian group X onto another such group Y. The solution of $F(x)=y$ is obtained by means of an iteration process and the degree of approximation of the successive iterants is estimated. *R. G. Bartle.*

Fenyő, István. Non-linear equations in Banach spaces. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 71–83 (1953). (Hungarian)

Main theorem: if f is a (non-linear) operator from a Banach space into another, with $f(x_0)=y_0$, and if f is Fréchet differentiable at x_0 with a continuous and (boundedly) invertible derivative, then the equation $f(x)=y$ has a unique solution for every y sufficiently near to y_0. The result is reformulated as a theorem of the implicit function type and also as a version of Newton's method; some applications to integral and to integro-differential equations are indicated. The theorem is offered as a generalization of a result of Kantorovič [Trudy Mat. Inst. Steklov. 28, 104–144 (1949); these Rev. 12, 419]. *P. R. Halmos.*

Ellis, David. A modification of the parallelogram law characterization of Hilbert spaces. Math. Z. 59, 94–96 (1953).

The author notes first that every Hilbert space can be made into a Banach lattice and that then the positive cone becomes a metric lattice under the definitions $\nu(x)=\|x\|^2$, $d(x,y)=\nu(x\vee y)-\nu(x\wedge y)$. Conversely, if this last condition holds in a Banach lattice and if the Jordan-von Neumann condition $\nu(x+y)+\nu(x-y)=2(\nu(x)+\nu(y))$ [Ann. of Math. (2) 36, 719–723 (1935)] holds for $x\geqq y\geqq 0$, then it holds for all x, y in B and B is a Hilbert space. *M. M. Day.*

Dixmier, J. Sur les bases orthonormales dans les espaces préhilbertiens. Acta Sci. Math. Szeged 15, 29–30 (1953).

Let $K=K_1+K_2+K_3$ be the orthogonal direct sum of the three Hilbert spaces K_i where $\dim K_1=\mathfrak{p}_1, \aleph_1\leqq\dim K_2=\mathfrak{p}_2\leqq\mathfrak{c}$, $\dim K_3=\aleph_0$. Then by choosing certain orthonormal subsets of K and special values for $\mathfrak{p}_1$ and $\mathfrak{p}_2$ the author shows the following. (a) For each cardinal $\mathfrak{m}>\aleph_0$, there is a pre-Hilbert space (a unitary space whose completion is a Hilbert space) whose completion has dimension $\mathfrak{m}$ and yet the pre-Hilbert space possesses no spanning orthonormal set. (b) There is a pre-Hilbert space in which every orthonormal set is countable, but whose completion has dimension $\mathfrak{c}$. (c) In every pre-Hilbert space whose completion has dimension greater than $\mathfrak{c}$, every maximal orthonormal system has a potency greater than $\mathfrak{c}$. *B. R. Gelbaum.*

Tagamlitzki, Y. Zur Geometrie des Kegels in den Hilbertschen Räumen. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie II. 47, 85–107 (1952). (Bulgarian. Russian and German summaries)

Let H be a (possibly incomplete) real Hilbert space. A non-void subset R of H is a cone if $tr\in R$ for all $r\in R$ and non-negative numbers t. A cone R is said to be ordinary if there exists $n\in H$ such that $(n,r)>0$ for $r\in R$, $r\neq 0$. A cone R is called strongly compact if every sequence $\{r_n\}_{n=1}^{\infty}$ of elements of R with bounded norms admits a subsequence converging strongly to an element of R. An element a of R is said to be irreducible with respect to R if a cannot be written as a linear combination of two linearly independent elements of R (i.e., if $b\in R$ and $a-b\in R$ imply that $a=\lambda b$ for some real λ). Fundamental theorem. Let R be an ordinary strongly compact cone and let K be a strongly closed convex cone such that $K\subset R$ and $K\neq R$. Then there exists an element $a\in R\cap K'$ which is irreducible with respect to R. The following stronger property is also established for a. There exist s and n in H such that $(n,x)>0$ and

$$\left\|\frac{a}{(n,a)}-s\right\|\geqq\left\|\frac{x}{(n,x)}-s\right\|$$

for all $x\in R$. This theorem is used to give new proofs of the Hausdorff moment theorem, Bernstein's representation theorem for completely monotone functions, and to establish two other theorems of the same general type. One of these asserts that if $f(x)$ is defined for $a\leqq x\leqq b$, if $f(x)\leqq f(a)$ for $a<x\leqq b$, if f has derivatives of all orders for $a<x\leqq b$, if $(-1)^k f^{(k)}(x)\geqq 0$ for $a<x\leqq b$ and $f^{(k)}(b)=0$ for $k=0,1,2,\cdots$, then $f(x)\equiv 0$. The method of proof is simple. For the Hausdorff moment theorem, for example, $H=$ all sequences $\{d_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty}a_n^2 2^{-n}<\infty$. The cone R is taken as all sequences $\{a_n\}_{n=0}^{\infty}$ for which $(-1)^k\Delta^k a_m\geqq 0$, $k,m=0,1,2,\cdots$. The cone K (clearly $\subset R$) is defined as all $\{\int_0^1 t^n d\alpha(t)\}_{n=0}^{\infty}$ for increasing $\alpha(t)$. It is shown that $K=R$ by proving that any vector irreducible with respect to R lies in K. *E. Hewitt* (Seattle, Wash.).

Hartman, Philip. On the essential spectra of symmetric operators in Hilbert space. Amer. J. Math. 75, 229–240 (1953).

The author proves certain theorems, familiar for differential operators, in abstract form. They concern the spectra of symmetric operators T with equal deficiency indices and those of their self-adjoint extensions A. To state some of these results, deduced for second order differential equations by Hartman and Wintner [Amer. J. Math. 72, 545–552 (1950); these Rev. 12, 29], let us adopt the author's notation. $\Sigma(A)$ denotes the spectrum of A, $\Pi(A)$ its point spectrum, $\Sigma'(A)$ its essential spectrum, $\Gamma(A)$ its continuous spectrum, $\Pi_0(T^*)$ the set of real $\lambda \in \Pi(T^*)$.

Then the following is proved.

$\mu \in \Pi(A)$ for some $A \leftrightarrow \mu \in \Pi_0(T^*) \leftrightarrow 0 \in \Pi(T-\mu I)(T^*-\mu I)$,
$\mu \in \Pi(A)$ for all $A \leftrightarrow \mu \in \Pi(T) \leftrightarrow 0 \in \Pi(T^*-\mu I)(T-\mu I)$,
$\mu \in \Sigma'(A)$ for some $A \leftrightarrow 0 \in \Sigma'(T-\mu I)(T^*-\mu I)$,
$0 \in \Sigma'(T^*-\mu I)(T-\mu I) \rightarrow \mu \in \Sigma'(A)$ for all A.

If the deficiency indices are not only equal but finite, then $\Sigma'(A)$ is independent of A. Every $\mu \in \Sigma(A)$ for some A. The principal lemma used in the proof of the theorems is the following: Let X be a closed operator with a domain $\mathfrak{D}(X)$ which is dense in H. Let $E(\lambda)$, $F(\lambda)$ be the resolutions of the identity belonging to X^*X, XX^*, respectively. Then there exists a unique bounded operator W with the properties that $W^*W = I - E(0)$ and $WW^* = I - F(0)$, and that if $0 \leqq \lambda \leqq \mu$ then $F(\mu) - F(\lambda) = W[E(\mu) - E(\lambda)]W^*$ and $E(\mu) - E(\lambda) = W^*[F(\mu) - F(\lambda)]W$. *František Wolf.*

Silberstein, J. P. O. On eigenvalues and inverse singular values of compact linear operators in Hilbert space. Proc. Cambridge Philos. Soc. 49, 201–212 (1953).

Let K be a compact (completely continuous) linear operator in a Hilbert space, $\{\kappa_n\}$ be the non-zero eigenvalues of K arranged in decreasing order of absolute value and let $\{\sigma_n\}$ be the inverse singular values of K likewise arranged (these are the values σ for which $(K^*K - \sigma^2 I)y = 0$ is satisfied by a non-trivial vector). Improving on work of Chang [Trans. Amer. Math. Soc. 67, 351–367 (1949); these Rev. 11, 523] the author shows that (i) if for some $\rho > 1$ the ratio $\sigma_n/\sigma_{[\rho n]}$ is bounded above, then $|\kappa_n| = O(\sigma_n)$ and (ii) if $\sigma_n = O(\mu_n)$, where the decreasing sequence $\{\mu_n\}$ satisfies the condition imposed on $\{\sigma_n\}$ in (i), then $|\kappa_n| = O(\mu_n)$. *B. Yood* (Eugene, Ore.).

Solomyak, M. Z. On characteristic values and characteristic vectors of a perturbed operator. Doklady Akad. Nauk SSSR (N.S.) 90, 29–32 (1953). (Russian)

Let A, A_0 be bounded linear self-adjoint operators on a Hilbert space, and λ_0 an isolated eigen-value of A_0 with multiplicity n. Let also $\|R_{\lambda_0}\| \cdot \|A - A_0\| < \frac{1}{2}$, where R_{λ_0} is the generalised resolvent at λ_0 of A_0. Then A will have n eigen-values in $|\lambda - \lambda_0| \leqq \|A - A_0\|$, not necessarily distinct. This fact and inequalities for the eigen-values and eigen-vectors are stated by the author, in extension of the (precise) inequalities given by Gavurin [same Doklady (N.S.) 76, 769–770 (1951); these Rev. 12, 617] for simple eigen-values, but under more general conditions than self-adjointness. Gavurin gave no proofs, and some of these (but not all) are now supplied. *F. V. Atkinson* (Ibadan).

Szőkefalvi-Nagy, Béla. Contributions to spectral theory by Hungarian mathematicians. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 85–100 (1953). (Hungarian)

A historical report concerned mostly with the works of F. Riesz and the author. *P. R. Halmos* (Chicago, Ill.).

Brudno, A. L. Norms of Toeplitz fields. Doklady Akad. Nauk SSSR (N.S.) 91, 11–14 (1953). (Russian)

Let $A = \{a_i^n\}$ be a Toeplitz matrix [Prace Mat.-Fiz. 22, 113–119 (1911)] and let A^n be the linear functional defined on the space (m) of bounded sequences by $A^n(x) = \sum a_i^n x_i$. For x in (m) define $|x| = \sup_i |x_i|$ and $\|x\| = \limsup_i \|x_i\|$. Similarly, define

$$|A| = \sup_n \textstyle\sum_i |a_i^n| \quad \text{and} \quad \|A\| = \limsup_n \textstyle\sum_i |a_i^n|.$$

Define $\mathfrak{A}$, the bounded field of A, to be the set of all x in (m) such that $\lim_n A^n(x)$ exists. Mazur and Orlicz [C. R. Acad. Sci. Paris 196, 32–34 (1933)] showed that if A and B are T-matrices for which $\mathfrak{A} \subseteq \mathfrak{B}$, then $\lim_n A^n(x) = \lim_n B^n(x)$ for every x in $\mathfrak{A}$: that is, the field $\mathfrak{A}$ determines a linear functional $\mathfrak{A}(\ )$ on A. This linear functional has the same norm whether regarded as an element of A^* or $(A/c_0)^*$. Theorem 1. There exists a in $\mathfrak{A}$ such that $|a| = \|a\| = 1$ and $\mathfrak{A}(a) = \|\mathfrak{A}\|$. Theorem 2. $\|\mathfrak{A}\|$ is the greatest lower bound of the set of numbers $\|B\|$ where B runs over the set of T-matrices with the bounded field $\mathfrak{A}$. It is remarked that A. S. Kronrod has an example to show the bound need not be attained; it is asked whether the bound is attained in the set of T-matrices B for which $\mathfrak{B} \supseteq \mathfrak{A}$. *M. M. Day.*

Brudno, A. L. Relative norms of Toeplitz matrices. Doklady Akad. Nauk SSSR (N.S.) 91, 197–200 (1953). (Russian)

In the notation of the preceding review let A and B be T-matrices and let δ be a non-negative real number. Define $N_A^B(\delta)$, the relative norm of B determined by A and δ, to be $\sup \|B(x)\|$ when $\|x\| \leqq 1$ and $\|A(x)\| \leqq \delta$. Theorem 4. $\mathfrak{B} \supseteq \mathfrak{A}$ if and only if $N_A^B(0) = 0$. Theorem 8 gives conditions, in terms of relative norms, on a sequence of T-matrices $\{A_i\}$ necessary and sufficient that there be a T-matrix A whose bounded field contains the union of the $\mathfrak{A}_i$. The paper concludes with an example of T-matrices not satisfying the condition. *M. M. Day* (Urbana, Ill.).

Lozinskiĭ, S. M. On the rapidity of convergence of a sequence of linear operations. Doklady Akad. Nauk SSSR (N.S.) 89, 609–612 (1953). (Russian)

Lozinskiĭ, S. M. On the rapidity of convergence of a sequence of linear trigonometric polynomial operations. Doklady Akad. Nauk SSSR (N.S.) 89, 785–787 (1953). (Russian)

The author continues his announcements without proofs of complicated theorems concerning linear operators on various classes of functions on $[0, 2\pi]$. [See Lozinskiĭ, same Doklady (N.S.) 64, 453–456 (1949); these Rev. 10, 529 and the literature there cited.] *E. Hewitt.*

Burgess, D. C. J. Tauberian theorems for abstract Dirichlet's series, with applications to the L^p spaces. Proc. London Math. Soc. (3) 3, 378–384 (1953).

This paper contains the abstract analogue of a theorem due to O. Szász [Trans. Amer. Math. Soc. 39, 117–130 (1936)]. The main result is: Let $\omega_s = \sum_1^\infty x_\nu e^{-\lambda_\nu s}$, $x_\nu \in X$, a B-space, the series being strongly convergent for every $s > 0$. Let

$$\text{(i)} \quad \lim_{s \to 0+} \omega_s = \gamma', \qquad \text{(ii)} \quad \sum_{\nu=1}^{n} \|x_\nu\|^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n)$$

as $n \to \infty$, $p > 1$ being fixed. Then $\sum x_\nu = \gamma'$. Condition (ii) may be replaced by $\|x_n\| = O[(\lambda_n - \lambda_{n-1})\lambda_n^{-1}]$ as $n \to \infty$. These results are applied to L_p-convergence of real Dirichlet series. Let the x_n be real numbers, suppose $\omega_s \in L_p(0, \infty)$ for

some $p \geqq 1$ and suppose that for some $q > 1$

$$\sum_{1}^{n} |a_\nu| \nu^{\lambda_\nu \varphi(1-1/p)} (\lambda_\nu - \lambda_{\nu-1})^{1-q} = O(\lambda_n).$$

Then the partial sums of ω_x converge in the mean of order p to ω_x in $(0, \infty)$. *E. Hille* (New Haven, Conn.).

Krasnosel'skiĭ, M. A., and Rutickiĭ, Ya. B. Differentiability of non-linear integral operators in Orlicz spaces. Doklady Akad. Nauk SSSR (N.S.) **89**, 601–604 (1953). (Russian)

The operator H defined by $H\varphi(x) = \int_G K(x, y) f(y, \varphi(y)) dy$ where G is a compact set in Euclidean n-space, f a non-linear function, is investigated via Orlicz spaces. Conditions are given under which the operator is (a) defined in some sphere of Orlicz space, (b) completely continuous, (c) differentiable (has a Fréchet differential), (d) asymptotically linear. No proofs are offered, but it is alleged that they are based upon results of the authors [Krasnosel'skiĭ, same Doklady (N.S.) **77**, 185–188; **79**, 389–392 (1951); Krasnosel'skiĭ and Rutickiĭ, ibid. **81**, 497–500 (1951); **85**, 33–36 (1952); these Rev. **12**, 836; **13**, 251, 357; **14**, 57; Rutickiĭ, Dopovidi Akad. Nauk Ukrain. RSR **1952**, no. 3 (unavailable)] and the results of Schauder and Leray. *B. Gelbaum.*

Nakano, Hidegorô. Linear topologies on semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 87–104 (1953).

The author gives a significant generalization of the Riesz-Fischer completeness theorem. Let R be a universally continuous semi-ordered linear space (conditionally complete vector lattice) of elements $f, g, \cdots$. A function $\lambda(u)$, defined for all $u \geqq 0$ in R and with $0 \leqq \lambda(u) < \infty$, is called a pseudo-norm if $\lambda(0) = 0$ and $u_1 \leqq u_2$ implies $\lambda(u_1) \leqq \lambda(u_2)$. Suppose given a family of pseudo-norms with the property: for each λ_α in the family, there is a constant $k(\alpha) < \infty$ and a finite number of pseudo-norms $\lambda_1, \lambda_2, \cdots, \lambda_n$ in the family such that for all u, v,

$$\lambda_\alpha(u+v) \leqq k(\alpha)[\max_i (\lambda_i(u)) + \max_i (\lambda_i(v))].$$

Now let f_μ be a net of elements in R with μ running over a directed set. Call f_μ a Cauchy net if for each λ in the given family, $\lambda(|f_\mu - f_\nu|) \to 0$; call f_μ convergent if for some f in R, for each λ in the given family, $\lambda(|f - f_\mu|) \to 0$. Every convergent net is Cauchy. The completeness theorem that every Cauchy net is convergent is established under the two assumptions: (1) if $u = \sup u_\mu$, then for each λ in the family $\lambda(u) = \sup \lambda(u_\mu)$; and (2) if for each λ in the family, $\sup \lambda(u_\mu)$ is finite, then $\sup u_\mu$ exists in R (it is supposed that $u_\mu \leqq u_\nu$ whenever $u \leqq \nu$). When only condition (1) is postulated, it is shown that every Cauchy net f_μ which is order-bounded (i.e., for some u in R, all $|f_\mu| \leqq u$) is convergent. (The reviewer notes that the proofs given are actually valid for the more general case when 'linear space' is replaced by 'group'.) A discussion is given of the uniform topology defined on R by means of pseudo-norms.

I. Halperin (Kingston, Ont.).

Nakano, Hidegorô. On transcendental points in proper spaces of discrete semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 105–110 (1953).

The author defines regular cardinals to be those cardinal numbers which are contained in every collection M with the properties: $\aleph_0 \in M$; $u \in M$ implies $2^u \in M$; $u_\alpha \in M$ for each α in a set J and if cardinal power of J is in M then $\sum u_\alpha \in M$; $u < v$ and $v \in M$ implies $u \in M$. He uses the technique of the theory of vector lattices to prove the following theorem: Let S be any set of points x, with cardinal power which is regular. Let R be the lattice of real-valued functions $f(x)$ partially ordered in the usual way. Then every linear positive functional $\varphi(f)$ has the form: $\varphi(f) = \sum c_i f(x_i)$ for suitable non-negative constants c_i and a suitable finite number of indices i, depending only on φ. A previous paper by Mackey [Bull. Amer. Math. Soc. **50**, 719–722 (1944); these Rev. **6**, 70] essentially pointed out that this result follows from the work of Ulam [Fund. Math. **16**, 140–150 (1930)]. *I. Halperin* (Kingston, Ont.).

Amemiya, Ichirô. A general spectral theory in semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 111–156 (1953).

The author considers the general semi-ordered linear space R and extends some results previously obtained by writers who assumed conditional or σ-conditional completeness. He defines the spectral space F as the set of functions $\varphi(x)$ with: for x in R, $\varphi(x) = 0, +\infty$ or $-\infty$; $\varphi(\lambda x) = \lambda \varphi(x)$ for $\lambda \neq 0$; $\varphi(x \cup y) = \max\{\varphi(x), \varphi(y)\}$; and $\varphi(x) \neq 0$ for some x. For each $a \neq 0$ in R, V_a = set of φ in F with $\varphi(a) \neq 0$ is shown to be non-empty: When the V_a are taken as a basis for the open sets, F becomes a T_0-space in which the V_a are the open-and-compact sets (related to Stone's imbedding of an abstract Boolean ring in a ring of subsets of a topological space).

Extensions, completions, subspaces and quotients of R are analysed from the point of view of spectral space. Nakano's theory of relative spectra is extended. R is isomorphic to a space of point-functions if and only if the spectral space satisfies a certain condition; R is isomorphic to a space of bounded point-functions if and only if R admits a function $\|a\|$ with: $0 < \|a\| < \infty$ for each $a \neq 0$; $\|\lambda a\| = |\lambda| \cdot \|a\|$; $|a| \leqq |b|$ implies $\|a\| \leqq \|b\|$; and

$$\|a \cup b\| = \max\{\|a\|, \|b\|\}$$

for positive a, b.

Compactification and extension theorems for point spaces due to Stone, Čech, and Hewitt are studied from the point of view of spectral space. A final chapter is devoted to such R which are also rings. *I. Halperin* (Kingston, Ont.).

Yosida, Kôsaku. An ergodic theorem associated with harmonic integrals. Proc. Japan Acad. **27**, 540–543 (1951).

Let R be an n-dimensional ($n > 1$), infinitely differentiable, orientable Riemann space, closed or open. Let H' be the set of infinitely differentiable exterior differential forms α on R vanishing outside a compact set. Let H be the Hilbert space obtained by completing H' under the norm $\|\alpha\| = (\alpha, \alpha)^{1/2}$ where $(\alpha, \beta) = \int_R \alpha \beta^*$, β^* being the adjoint form of β. Let $d\alpha$ and $\delta\alpha$ be the exterior differential and codifferential of $\alpha \in H'$, $\Delta = d\delta + \delta d$ the Laplacian. The author proves that starting with any $\alpha \in H$ one may obtain its harmonic part by a stochastic process. Denoting by $\bar{\Delta}$ the Friedrichs-Freudenthal self-adjoint, non-positive definite extension of Δ, he considers the diffusion equation $Y'(t) = \bar{\Delta}[Y(t)]$ with the initial condition $\lim_{t \to +0} \|Y(t) - \alpha\| = 0$, $\alpha \in D[\bar{\Delta}]$, and proves that Δ generates a semi-group T_t, strongly continuous for $t \geqq 0$, such that $Y(t) = T_t \alpha$. The ergodic theorem which he proves asserts the existence of the strong limit of $T_t \alpha = T_\infty \alpha$ as $t \to +\infty$. Here $\Delta T_\infty \alpha = 0$ and $(\alpha - T_\infty \alpha, \beta) = 0$ for every β such that β and $\Delta\beta$ belong to H and $\Delta\beta = 0$. If R is euclidean or a closed space, Δ may be taken as the smallest closed extension $(\Delta')'$ of Δ. The elliptic character of Δ is used in proving infinite differentiability of the various forms involved. *E. Hille* (New Haven, Conn.).

Ura, Taro. Sur les courbes définies à la surface du tore par des équations admettant un invariant intégral. Ann. Sci. Ecole Norm. Sup. (3) **69**, 259–275 (1952).

In this paper a study is made of flows on the surface of a torus, under the assumptions that a positive invariant integral exists and that there exist a finite number of singular points of saddle-point or center type. The term "periodic" is broadened to include trajectories through saddle-points and a flow is termed "ergodic" if every trajectory is everywhere dense. The following theorems are proved. 1. If there is one periodic trajectory not homotopic to 0, then all trajectories are periodic. 2. If there is no periodic trajectory not homotopic to 0, then the flow in the torus minus the set of periodic trajectories homotopic to 0 is ergodic (misprinted as "périodique"). Special results are given for the case of one saddle-point and one center. *W. Kaplan.*

Okstobi, D. [Oxtoby, J.] Ergodic sets. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 75–97 (1953). (Russian)

Translated from Bull. Amer. Math. Soc. **58**, 116–136 (1952); these Rev. **13**, 850.

Theory of Probability

***Braithwaite, R. B. The meaning of empirical probability statements.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 136–138. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.

Medgyessy, Pál. Sur quelques problèmes en relation avec la planche de Galton. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 165–174 (1953). (Hungarian. Russian and French summaries)

This is a discussion of demonstrations which can be carried out on a Galton board. The board is modified by providing a movable row of receptacles which can be opened singly. *E. Lukacs* (Washington, D. C.).

Risser, R. Note relative aux tirages contagieux. Bull. Trimest. Inst. Actuaires Français **50**, 235–258 (1951).

The Eggenberger-Pólya urn scheme illustrating a contagious process (sampling with replacement of the ball chosen and Δ more of the same color) is extended to 3 colors (which, it is noted, makes evident the generalization to n colors). The limiting forms for large samples, which are multi-variable normal, are developed. *J. Riordan.*

Greenwood, R. E., and Gleason, A. M. Distribution of round-off errors for running averages. Pacific J. Math. **3**, 605–611 (1953).

Les auteurs considèrent une suite de variables aléatoires $G_1, G_2, \cdots, G_n, \cdots$ entières positives et les moyennes mobiles S_n définies par: $kS_{n+1}=(k-1)S_n+G_{n+1}$ (k entier>0); les S_n ne sont pas entières en général; en leur substituant des valeurs entières approchées T_n, on commet des erreurs E_n définies par: $kE_{n+1}=(k-1)E_n+D_n$, où D_n est une correction dont la valeur aléatoire résulte des G_j; dans deux cas simple où $k=2$, les auteurs montrent que la distribution de E_n tend vers une limite pour $n\to+\infty$ et donnent cette limite.

R. Fortet (Paris).

Prohorov, Yu. V. Asymptotic behavior of the binomial distribution. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 135–142 (1953). (Russian)

Let

$$B(x)=\binom{n}{x}p^x(1-p)^{n-x},\quad a=np,\quad \sigma=(npq)^{1/2},\quad u=(x-a)/\sigma,$$

and

$$\Pi_1(x)=0,\quad x<0;\quad \Pi_1(x)=\frac{a^x}{x!}e^{-a},\quad x\geqq 0;$$

$$\Pi_2(x)=(2\pi npq)^{-1/2}e^{-\frac{1}{2}u^2};$$

$$\Pi_3(x)=0,\quad x<0;\quad \Pi_3(x)=\frac{(n-a)^x}{x!}e^{-(n-a)},\quad x\geqq 0;$$

$$\rho_k(p,n)=\sum|B(x)-\Pi_k(x)|\quad (k=1,2,3),$$

$$\lambda_1=\frac{2}{\sqrt{(2\pi)}}e^{-1/2},\quad \lambda_2=\frac{1}{3\sqrt{(2\pi)}}(1+4e^{-3/2}),$$

$$\lambda=\lambda_1^{1/3}\lambda_2^{2/3}.$$

The author proves the following theorems.

Theorem 1: As $n\to\infty$,

$$\sup_{0\leq p\leq 1}\min_k \rho_k(p,n)=\lambda n^{-1/3}+O(n^{-2/3}).$$

Theorem 2: $\rho_1(p,n)=\lambda_1 p+pO(\min(1,(np)^{-1/3}))$.

Theorem 3: $\rho_2(p,n)=|q-p|\sigma^{-1}\lambda_2+O(\sigma^{-2})$.

J. Wolfowitz (Ithaca, N. Y.).

Blanc-Lapierre, A. Notes on Poisson distributions. Revista Unión Mat. Argentina **15**, 3–6 (1951). (Spanish)

The author studies a generalized Poisson distribution and characterizes it in terms of the conditional probability that an event will occur between t_2 and t_2+dt_2, provided the preceding event has occurred at t_1. *E. Lukacs.*

Fisz, M. The limiting distribution of the difference of two Poisson random variables. Zastosowania Mat. **1**, 41–45 (1953). (Polish. Russian and English summaries)

The statement is proven that if the random variables X_1 and X_2 are independent and have Poisson distributions with $E(X_1)=\lambda_1$, $E(X_2)=\lambda_2$, then their standardized difference $X=[(X_1-X_2)-(\lambda_1-\lambda_2)]/(\lambda_1+\lambda_2)^{\frac{1}{2}}$ has asymptotically distribution $N(0,1)$ as $\lambda_1\to+\infty$ and $\lambda_2\to+\infty$. This is obvious since then X_1 and X_2 are each asymptotically normal. The author could have concluded that already $\lambda_1+\lambda_2\to+\infty$ is sufficient for X having asymptotically distribution $N(0,1)$. Of interest is a numerical tabulation for $\lambda_1=2$, $\lambda_2=1$, given in the paper, which suggests that even for different and fairly small λ_1, λ_2 the normal approximation is good. *Z. Birnbaum* (Seattle, Wash.).

Consael, R. Sur les processus composés de Poisson à deux variables aléatoires. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **27**, no. 6, 44 pp. (1952).

The compound Poisson process treated by Lundberg [Thesis, Uppsala, 1940; these Rev. **2**, 230] is generalized to two dimensions. Let the probability of n events of the first category and m of the second category in the interval $(0,t)$ be given by

$$P_{n,m}(t)=\int_0^\infty\int e^{-t(x+y)}\frac{(tx)^n}{n!}\cdot\frac{(ty)^m}{m!}dU(x,y),$$

where U is a bivariate probability distribution. Expressions are given for the generating function and moments of m and n. Various differential equations are given and solved relating the functions $P_{n,m}(t)$ and the intensity functions $p_{n,m}(t)$ and $q_{n,m}(t)$, where $p_{n,m}(t)\Delta t$ is approximately the probability of a first-category event between t and $t+\Delta t$, and $q_{n,m}(t)$ is defined similarly for second-category events. Marginal and conditional distributions are found. Various special cases and applications are considered, including the relation of these processes to Neyman's Type A distribution studied in connection with contagion [Ann. Math. Statistics 10, 35–57 (1939), pp. 48–53]. *T. E. Harris.*

Hammersley, J. M. On counters with random dead time. I. Proc. Cambridge Philos. Soc. **49**, 623–637 (1953).

The counter problem discussed in the first part of the paper is equivalent to the study of the distribution of the number of gaps between intervals (recorded events) of random length placed randomly on a circle. The author determines the moments of this distribution and proves also that it is asymptotically normal. In the second part of the paper the concept of paralysis is introduced. It is assumed that every start of a recorded impulse is also the start of an impulse (of constant or random duration) paralyzing the counter. Self-paralyzing mechanisms can be used to reduce the tendency of the counter to register the same impulse more than once. Hence this modification of the counter problem is of considerable practical interest. The first two moments of the distribution of the number of gaps are computed, this distribution is again asymptotically normal. In case the recorded impulses are of constant length, the moments may be expressed in terms of the "tapered exponential function" defined by

$$e^*(a, b) = 1 + \sum_{k=1}^{\infty} \{\tfrac{1}{2}a(b-k+|b-k|)\}^k/k!.$$

Tables of $e^*(a, b)$ are given for $a = 0.00$, (0.05), 0.35, 0.3679 and $b = 0.0$, (0.5), 5.0, (1.0), 10.0. *E. Lukacs.*

Fisher, Ronald. Dispersion on a sphere. Proc. Roy. Soc. London. Ser. A. **217**, 295–305 (1953).

Soient une sphère Σ de centre 0, de rayon 1, et Ω un point de Σ; on choisit au hasard sur Σ, indépendamment, selon la même loi de probabilité, N points $M_1, \cdots, M_i, \cdots, M_N$; soit $\overline{S} = \sum_{i=1}^{N} \overline{0M_i}$; l'auteur étudie la distribution de $|\overline{S}|$ et de l'angle $\overline{S}$ avec $\overline{0\Omega}$: a) lorsque la densité de probabilité de M_i sur Σ est constante; b) lorsque cette densité est proportionnelle à $e^{K\cos\theta}$ [$\theta = (\overline{0\Omega}, \overline{0M_i})$]. Dans le cas b), l'auteur traite quelques problèmes statistiques: estimation de K si on connait Ω, si on connait seulement la direction de 0Ω, test de signification, etc.; les résultats sont applicables au cas de visées comportant une dispersion notable; exemples numériques. *R. Fortet* (Paris).

Rényi, Alfréd. On projections of probability distributions. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 59–69 (1953). (Hungarian)

The author considers the following theorem of Radon: if K is a bounded domain in the plane and if f is a continuous function on K such that the integral of f vanishes on every chord of K, then f is identically zero. He reformulates and generalizes the theorem and shows thereby that it can be easily derived from the theorem of Cramér and Wold: Every planar probability distribution is uniquely determined by its linear projections. The paper concludes with some analogous combinatorial results concerning the unique determination of mass distributions concentrated at a finite set. *P. R. Halmos* (Chicago, Ill.).

Berger, Agnes. On orthogonal probability measures. Proc. Amer. Math. Soc. **4**, 800–806 (1953).

Two sets of probability measures (with the same domain) are called orthogonal if there exists a set that has measure zero for all the measures of one set and measure one for all the measures of the other set. The main purpose of the paper is to give conditions under which two sets of infinite power measures are orthogonal. A typical result is that if one set of measures is σ-compact and if another set of measures has the Lindelöf property (both topological hypotheses interpreted in the weak topology of the set of all probability measures with a fixed domain), then the set of all infinite powers of the first set is orthogonal to the set of all infinite powers of the second set. *P. R. Halmos.*

Robbins, Herbert. On the equidistribution of sums of independent random variables. Proc. Amer. Math. Soc. **4**, 786–799 (1953).

Let S_n be the sum of n mutually independent random variables with a common distribution. (The following statements assume that this distribution is not confined to integral multiples of some number; in the latter case the results are modified in an obvious way.) Then for a large class of functions h whose integral means $M(h)$ on $(-\infty, \infty)$ exist, the author proves that $\lim_{n\to\infty} (1/n)\sum_1^n h(S_i) = M(h)$ with probability 1. It follows, among other results, that, if I is any interval, of length c, the S_i's, when reduced modulo c to lie in I, are equidistributed in I, with probability 1. These results are extended to random variables with values in a compact topological group in a section attributed to Kakutani. Corresponding results are obtained in the continuous-parameter case, in which S_n is replaced by S_t and t is any positive real number. *J. L. Doob.*

Udagawa, Masatomo. Asymptotic properties of distributions of some functionals of random variables. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **2**, no. 2–3, 1–98 (1952).

The paper consists of three more or less unrelated parts. The first part (Chapter I) is devoted to slight improvements and extensions of the theory of independent functions in an infinite interval [cf. Kac and Steinhaus, Studia Math. **7**, 1–15 (1938); Hartman, van Kampen and Wintner, Amer. J. Math. **61**, 477–486 (1939)]. Section 4 of Chapter I treats an entirely different topic. A typical result is that if $X(t)$ is a stationary process (wide sense) with $E(X(t)) = 0$, $E(X^2(t)) = 1$ and if $F(x)$ is its spectral function with a pure point spectrum $\{\lambda_k\}$ satisfying the condition

$$\inf\{\lambda_k - \lambda_{k-1}\} > 0,$$

then with probability 1 the sample functions are almost periodic S^2. (This as the author points out is weaker than a theorem of Slutsky [Actualités Sci. Ind., no. 738, Hermann, Paris, 1938, pp. 33–55] but the author admits that he does not understand Slutsky's proof.) Chapters II and III are devoted to a generalization of an "invariance principle" first introduced by Erdös and the reviewer as a method of proving certain limit theorems. The author follows closely the general treatment of Donsker [Mem. Amer. Math. Soc., no. 6 (1951); these Rev. **12**, 723], but weakens the hypotheses so as to include the case of random variables belonging to the domain of attraction of an

arbitrary symmetric stable law. Chapter IV deals with some analytic properties of Wiener measure. A typical result is that if $x_0'(t) \in L^2(0, 1)$ there exists a real λ_0 such that the Wiener integral of $\exp(\lambda\|x-x_0\|^2)$ (the norm being the Hilbert space norm) is finite for all $\lambda<\lambda_0$. Extensive use is made of the work of Cameron and Martin [e.g., Ann. of Math. (2) **45**, 386–396 (1944); these Rev. **6**, 5].

M. Kac (Ithaca, N. Y.).

Marušin, M. N. On necessary and sufficient conditions for applicability of a limit theorem of order $p<2$. Doklady Akad. Nauk SSSR (N.S.) **90**, 727–730 (1953). (Russian)

The author proves a version of the central limit theorem stated without proof by Bernstein [same Doklady (N.S.) **24**, 3–7 (1939); these Rev. **1**, 340]. The theorem gives necessary and sufficient conditions that, if $0<p<2$, if s_n is the sum of n independent random variables, and if D_n^2 is the variance of the sum of these variables after each is cut off at some suitable point, then s_n/D_n is asymptotically normal and $E\{|s_n/D_n|^p\}$ is asymptotically the corresponding normal moment. *J. L. Doob* (Urbana, Ill.).

van Dantzig, D. Another form of the weak law of large numbers. Niew Arch. Wiskunde (3) **1**, 129–145 (1953).

Let $\{x_{ij}\}$, $j=1, \cdots, n_i$; $i=1, 2, \cdots$ ad inf., be a sequence of chance variables such that all chance variables with the same first index are independent. Let $S_i=\sum_{j=1}^{n_i} x_{ij}$, and let $F_{ij}(x)$ be the distribution function of x_{ij}. Let $b=b_1, b_2, \cdots$ ad inf. be a given sequence of positive numbers. The chance variables $\{x_{ij}\}$ are said to obey the law of large numbers for the sequence b if there exists a sequence $c=c_1, c_2, \cdots$ ad inf. such that, for any $a>0$,

$$\lim_{i\to\infty} P\{|S_i-c_i|>ab_i\}=0$$

where $P\{\ \}$ denotes the probability of the relation in braces. Let

$$\gamma_{ij}(g, h)=\int \min(1, y^2)dF_{ij}(g+hy), \quad \gamma_{ij}(h)=\inf_g \gamma_{ij}(g, h).$$

The author proves that a necessary and sufficient condition for the chance variables $\{x_{ij}\}$ to obey the law of large numbers for the sequence b is that

$$\lim_{i\to\infty} \sum_{j=1}^{n_i} \gamma_{ij}(b_i)=0.$$

This result applies with very little change to the case where the chance variables $\{x_{ij}\}$ are vectors in m-space. It is sufficient to consider y^2 as the square of the length of the vector y. *J. Wolfowitz* (Ithaca, N. Y.).

Harris, T. E., and Robbins, Herbert. Ergodic theory of Markov chains admitting an infinite invariant measure. Proc. Nat. Acad. Sci. U. S. A. **39**, 860–864 (1953).

The authors show how to apply the point ergodic theorem for spaces of infinite measure to (integer-valued parameter) Markov processes possessing infinite-valued stationary absolute "probability distributions". As an application, they prove that, if s_n is the sum of n mutually independent random variables with a common distribution, and if the sequence $\{s_n, n\geq 1\}$ is everywhere dense with probability 1, then, if h and k are Lebesgue integrable over $(-\infty, \infty)$, with integrals $\bar{h}$, $\bar{k}$, $\bar{k}\neq 0$, it follows that $\lim_{n\to\infty} \sum_1^n h(a+s_j)/\sum_1^n k(a+s_j)=\bar{h}/\bar{k}$ with probability 1, for almost all a (Lebesgue measure). Moreover, for every a, this limit equation is true with probability 1 simultaneously for every h, k under the additional restriction that these functions be Riemann integrable and vanish in a neighborhood of infinity. If the s_j's are integer-valued, an analogous result is obtained, in which the Lebesgue measure of a linear set is replaced by the cardinal number of a set of integers. *J. L. Doob* (Urbana, Ill.).

Riordan, John. Addendum to Delay curves for calls served at random. Bell System Tech. J. **32**, 1266 (1953).

See same J. **32**, 100–119 (1953); these Rev. **14**, 664.

Mathematical Statistics

***Graf, Ulrich, und Henning, Hans-Joachim. Formeln und Tabellen der mathematischen Statistik.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. vi+102 pp. DM 9.

This book is a compilation of formulae and tables used in mathematical statistics. Examples are worked out to show how the formulae may be applied and the tables are supplemented by some nomographs. Most of the formulae correspond to rather elementary results. While applications appear involving analysis of variance, control charts and sequential analysis, nothing especially deep appears. This book will be most useful to applied statisticians who do not have a very large variety of problems to deal with.

While the book was intended to be of limited size, it would have been appropriate for it to have more on nonparametric tests. *H. Chernoff* (Stanford, Calif.).

Olekiewicz, M. An extended table of Student's t-distribution for one-sided and two-sided tests of significance at 5 and 1% probability levels. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **5** (1951), 161–163 (1953). (Polish and Russian summaries)

A table of Student's t is given for one-sided and two-sided tests of significance at 5% and 1% probability levels. The degrees of freedom covered are 1(1)30(2)100; 120 and ∞.

H. Chernoff (Stanford, Calif.).

Olekiewicz, M. Tables of expected values and variances of numbers of runs in random sequences with probabilities of exceeding expected values. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **5** (1951), 147–159 (1953). (Polish and Russian summaries)

Let R equal the number of runs observed in a sequence of n observations each of which may take one of two values. If one of these values occurs less frequently than the other let $f_{\min}$ represent the number of occurrences of the first and $f_{\max}$ the number of occurrences of the second. Given $f_{\min}$ and $f_{\max}$, one may obtain $E(R)$, the conditional expectation of R, $D^2(R)$, the conditional variance of R, and $P[R>E(R)]$, the conditional probability that R exceeds $E(R)$. Tables are given for $E(R)$, $P(R>E(R))$, $D^2(R)$ and $D^2(R/n)$ for $1\leq f_{\min}\leq f_{\max}\leq 20$. An example is given indicating how the tables of $P(R>E(R))$ may be used to combine the results of a number of independent experiments. *H. Chernoff.*

Gordon, Mordecai H., Loveland, Edward H., and Cureton, Edward E. An extended table of chi-square for two degrees of freedom, for use in combining probabilities from independent samples. Psychometrika 17, 311–316 (1952).

A table of values of Chi-square for two degrees of freedom corresponding to values of P from .001 to .999 is presented together with a description and an example of its use in combining significance levels from two or more independent samples. The table is a trivial modification of a natural log table since the Chi-square value is $-2 \log P$. The authors are somewhat ambiguous in their discussion of how to apply the procedure. It might diminish the ambiguity if it were pointed out that for studies which yield results in the opposite direction from the majority the P values should be greater than 1/2. It should be pointed out that this method should be applied mainly to studies of comparable power.

H. Chernoff (Stanford, Calif.).

Fog, David. Contingency tables and approximate χ^2-distributions. Math. Scand. 1, 93–103 (1953).

The author considers contingency tables where the marginal frequencies are given. The usual expression for the chi-square statistic which is used to test for independence is shown to be a sum of squares of normalized and mutually orthogonal linear forms. Those forms are uncorrelated with zero means and approximately unit variance. The variates in these forms are of the form $(n-v)/v^{1/2}$ where n is the observed frequency in a given cell and v the expected frequency in that cell. No special interpretation is given to these linear forms. The standard proof in the literature of the asymptotic distribution of the chi-square statistic almost yields this result. *H. Chernoff* (Stanford, Calif.).

Vincze, István. Sur la détermination du coefficient de régression. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 1 (1952), 215–237 (1953). (Hungarian. Russian and French summaries)

Let y be a random variable and assume that (1) the regression of y on x is linear, (2) the random variable y can be observed at any one of s fixed values $x_1, \cdots, x_s$ of the (non-random) variable x, (3) the y's measured at x_i are independently and identically distributed with known variance σ_i^2. The total number n of observations is fixed but may be distributed in an arbitrary manner, say n_i observations at x_i, among the s possible values of x. The author investigates the problem of selecting the n_i ($i=1, 2, \cdots, s$) in such a manner that the variance of the estimate of the regression coefficient is minimized. *E. Lukacs.*

Sukhatme, P. V. Measurement of observational errors in surveys. Rev. Inst. Internat. Statistique 20, 121–134 (1952).

This paper discusses a special case of the model considered in a paper by the author and G. R. Seth [J. Indian. Soc. Agric. Statistics 4, 5–41 (1952); these Rev. 14, 390].

D. M. Sandelius (Uppsala).

Lah, Ivo. Contribution to the calculus of the mathematical expectation. Sankhyā 12, 247–264 (1953).

In this paper which harks back to an earlier day in mathematical statistics, the author gathers together a number of expected values, none really new, of p^r and $p^r(1-p)^s$ in which p is the observed success ratio in a sample of n from a binomial universe. With some extension to the bivariate case, he carries out the formal expansion of the expected value of the mean square contingency, the coefficient of correlation, and intercept and slope in a bivariate regression, in the manner of Tschuprow, to an additional power or two of $1/n$. *C. C. Craig* (Ann Arbor, Mich.).

Hayashi, Chikio, and Akaike, Hirotugu. On a matching problem. Ann. Inst. Statist. Math., Tokyo 4, 55–64 (1953).

From a population of $2N$ individuals divided into R strata N pairs of individuals are drawn at random without replacement. The distribution of the number of pairs consisting of individuals from the same stratum is studied. An explicit expression is given for $R=2$. The results are applied to sociometric data. *D. M. Sandelius* (Uppsala).

Ogawa, Junjiro. On the sampling distributions of classical statistics in multivariate analysis. Osaka Math. J. 5, 13–52 (1953).

The paper gives a good systematic presentation of the classical distributions in normal k-variate sampling theory, in particular the distributions of T^2, $r_{1(2\cdots k)}$, $r_{12\cdot 3\cdots k}$, and Bartlett's decomposition theorem. The method is that of conditional distributions: for fixed values of $k-1$ or $k-2$ of the variates, the distributions of the statistics in question turn out to be independent of the conditioning variables, or to depend only on a relatively simple function of them. [For the central case (i.e., the case where the $m_i=0$, $\rho_{1(2\cdots k)}=0$, $\rho_{12\cdot 3\cdots k}=0$, respectively), cf. Elfving, Skand. Aktuarietidskr. 30, 56–74 (1947); these Rev. 9, 48. The unified treatment of the non-central case is new.]

G. Elfving (Helsingfors).

Des Raj. On Mill's ratio for the type III population. Ann. Math. Statistics 24, 309–312 (1953).

Let the probability density for a standardized Type III distribution be $Cf(X)$ for $-2/\alpha_3 \leqq X < \infty$ where

$$f(X) = [1+\tfrac{1}{2}\alpha_3 X]^\beta, \quad \beta = 4\alpha_3^{-2} - 1, \quad 0 < \alpha_3 \leqq 2,$$

and let $G(x) = \int_x^\infty f(X)dX$. The author proves that the ratio $\mu(x) = f(x)/G(x)$ satisfies the inequalities

$$(x+\tfrac{1}{2}\alpha_3)(1+\tfrac{1}{2}\alpha_3)^{-1} \leqq \mu(x) \leqq 2\{-x+[x^2+4(1+\tfrac{1}{2}\alpha_3 x)]^{\frac{1}{2}}\}^{-1}.$$

Z. W. Birnbaum (Seattle, Wash.).

Moshman, Jack. Critical values of the log-normal distribution. J. Amer. Statist. Assoc. 48, 600–609 (1953).

Rider, Paul R. The distribution of the product of ranges in samples from a rectangular population. J. Amer. Statist. Assoc. 48, 546–549 (1953).

Kitagawa, Tosio, Kitahara, Teisuke, Nomachi, Yukio, and Watanabe, Nobuo. On the determination of sample size from the two sample theoretical formulation. Bull. Math. Statist. 5, no. 3–4, 35–45 (1953).

The standard confidence region of confidence α derived from a sample of fixed size n for the mean of a normal distribution with unknown variance σ^2 has length proportional to s the standard deviation of the sample and inversely proportional to $n^{1/2}$. Hence no fixed sample size assures one a probability exceeding $1-\beta$ of obtaining a length less than any prescribed number d. Accordingly, a two-sample scheme is proposed where n_2 the second sample size is a random variable determined by the data of first sample so as to assure a probability of $1-\beta$ of obtaining a length less than d. Tables are given indicating the distribu-

tion of n_2 given α, β, $d\sigma^{-2}$ and n_1 the sample size of the first sample. *H. Chernoff* (Stanford, Calif.).

Bartlett, M. S. Approximate confidence intervals. Biometrika **40**, 12–19 (1953).

Asymptotic confidence intervals for an unknown parameter may be obtained directly from $\partial L/\partial\theta$, the derivative of the log of the likelihood by making use of its asymptotic normality with mean 0 and variance equal to the information. The author proposes a second approximation which tends to eliminate the effect of the skewness of the distribution of $\partial L/\partial\theta$. This second approximation is compared with the first (the usual) approximation and exact values for the variance of a normal distribution and the mean of a Poisson. A third example involving tracks of cosmic ray particles which decay according to an exponential distribution is also considered.

The extension to confidence regions involving more than one parameter is said to be straightforward and is to be discussed in a further paper, but the case of one unknown parameter plus "nuisance parameters" is said to be more subtle. *H. Chernoff* (Stanford, Calif.).

Aroian, Leo A., and Levene, Howard. The effectiveness of quality control charts. J. Amer. Statist. Assoc. **45**, 520–529 (1950); erratum **47**, 685 (1952).

The authors derive spacing and effectiveness functions of quality control charts. These functions are associated with what is called a "continuing test." Some principles of Neyman-Pearson theory are extended to this type of statistical test. *D. F. Votaw, Jr.* (New Haven, Conn.).

Epstein, Benjamin, and Tsao, Chia Kuei. Some tests based on ordered observations from two exponential populations. Ann. Math. Statistics **24**, 458–466 (1953).

On the basis of the r_i smallest of n_i observations on the random variable X_i, defined in (A_i, ∞) and with p.d.f. $f(x_i, A_i, \theta_i) = \theta_i^{-1} \exp[-(x_i - A_i)/\theta_i]$, $i = 1, 2$, likelihood ratio tests are obtained for the hypotheses H_1: $\theta_1 = \theta_2$, given A_1, A_2; H_2: $\theta_1 = \theta_2$, given only that $A_1 = A_2$; H_3: $\theta_1 = \theta_2$; H_4: $A_1 = A_2$, given θ_1, θ_2; H_5: $A_1 = A_2$, given only that $\theta_1 = \theta_2$; H_6: $A_1 = A_2$; H_7: $A_1 = A_2$ and $\theta_1 = \theta_2$. For all tests, except the one for H_7, equivalent tests distributed as F or χ^2 are found. A table of the corresponding critical regions is given. *D. M. Sandelius* (Uppsala).

Sandelius, Martin. Some unbiased estimates for a type of two-phase sampling. Kungl. Lantbrukshögskolans Annaler **19**, 113–126 (1953).

A random sample is taken from a population π of known size $N > 0$ which consists of strata π_i of unknown sizes $N_i \geqq 0$, $i = 1, \cdots, k$. From the sample individuals thus obtained from π_i, a sub-sample of size n_i is taken, the n_i being determined by a rule agreed upon before any sampling was undertaken. For the individuals obtained by this two-phase sampling the values of a variate X are recorded. A very general procedure is described for constructing, from this information, unbiased estimates of the mean of X in π and of the variances of such estimates. Interesting special cases are discussed and an application is described.

Z. Birnbaum (Seattle, Wash.).

Basu, D. On a class of admissible estimators of the normal variance. Sankhyā **12**, 57–62 (1952).

The estimators in question for quadratic weight function and fixed sample-size n are of the form $aS+b$ for certain values a, b, where S is the sample sum of squares about the population or sample mean, depending on whether or not the former is known. The methods are those of Bayes solutions and a modification of Blyth's [Ann. Math. Statistics **22**, 22–42 (1951); these Rev. **12**, 622], proofs being rather sketchy in the latter case. *J. Kiefer* (Ithaca, N. Y.).

Fraser, D. A. S. The Behrens-Fisher problem for regression coefficients. Ann. Math. Statistics **24**, 390–402 (1953).

Let U_α $(\alpha = 1, \cdots, m)$ be independent normally distributed random variables with common variance σ^2 and means $E(U_\alpha) = \sum_1^p x_{i\alpha}\beta_i + \sum_1^q y_{j\alpha}\tau_j$ where the x, y are known numbers, the σ, β_i, τ_j unknown parameters. Let U_α^* $(\alpha = 1, \cdots, n > m)$ be another set of variables, independent of the former one and with the same properties, the parameters and regression variables being now σ^*, $x^*_{i\alpha}$, etc. The problem is to test the hypothesis $\tau_1^* = \tau_1, \cdots, \tau_q^* = \tau_q$. In the case $q = 1$, the author generalizes the solutions given by Scheffé for $p = 0$ and by Barankin for $p = 1$; the outcome is again a t-test. The main new contribution is the proof that the test is most powerful at least within a very large class of tests based on similar regions. In the cases $q > 1$, no such class is known. An asymptotically best, and a plausible exact test are discussed. *G. Elfving* (Helsingfors).

Anscombe, F. J. Sequential estimation. J. Roy. Statist. Soc. Ser. B. **15**, 1–21; discussion, 21–29 (1953).

Let $U_1, U_2, \cdots$ be independent, identically distributed chance variables with a finite fourth moment and a distribution which has an absolutely continuous component. A sequence of functions (called boundaries) $f_1(n), f_2(n), \cdots$ is considered given. Each $f_r(n)$ is defined for all positive integers, and $f_r(n)$ is subject to conditions on its growth as r and n approach infinity which are difficult to describe here. Let N_r be the least integer n such that $\sum_{i=1}^n U_i \leqq f_r(n)$ (for some functions f_r the sign in the preceding inequality is reversed). The author studies the asymptotic (with r) distribution of N_r. When the latter is suitably normalized the asymptotic distribution is shown to be normal. The author obtains an additional term of its expansion into an asymptotic series and the principal part of several of its moments. The results are then applied to problems of statistical estimation, in particular to those of estimating sequentially the mean of a normal distribution with unknown variance and the parameters of a birth and death process.

J. Wolfowitz (Ithaca, N. Y.).

Kudô, Akio. Note on the estimation of the mean value of the stochastic process. Bull. Math. Statist. **5**, no. 3–4, 53–58 (1953).

Let $\{x(t)\}$, $0 \leqq t \leqq T$, be a stationary stochastic process with $Ex(t)$ constant. The author obtains the variances (in one case the asymptotic variance) of the following unbiased estimators of $Ex(t)$. (1) $N^{-1}\sum_{i=1}^N x(t_i)$, where $t_1, \cdots, t_N$ are chance variables independently and uniformly distributed over the interval $(0, T)$. (2) $N^{-1}\sum_{i=1}^N x(t_i)$, where $t_1, \cdots, t_N$ are the time points at which the sample function of a continuous time parameter Poisson process with constant density has saltuses within the interval $[0, T]$; the estimator is conditional upon $N \geqq 1$. (3) $T^{-1}\sum_{k=1}^{N+1}(t_k - t_{k-1})x(\xi_k)$, where $t_0 = 0$, $t_{N+1} = T$, N and $t_1, \cdots, t_N$ are the chance variables defined in (2) above, and $\xi_1, \cdots, \xi_N$ are independent chance variables, with ξ_k $(k = 1, \cdots, N)$ distributed uniformly in the interval (t_{k-1}, t_k). *J. Wolfowitz* (Ithaca, N. Y.).

Inada, Ken-iti. On a certain decision problem under some constraints. Ann. Inst. Statist. Math., Tokyo **4**, 65–82 (1953).

The author gives a number of examples, principally from statistics, where he obtains a minimax solution for the statistician (player 1), the mixed strategies of Nature (player 2) being subject to several constraints and the pure strategies being finite in number. *J. Wolfowitz.*

Epstein, Benjamin, and Sobel, Milton. Life testing. J. Amer. Statist. Assoc. **48**, 486–502 (1953).

Observations from the density function $f(x;\theta)=\theta^{-1}e^{-x/\theta}$, $\theta>0$, $x>0$, become available in order

$$x_{1,n}\leqq x_{2,n}\leqq\cdots\leqq x_{r,n}\leqq\cdots\leqq x_{n,n}.$$

Experimentation is discontinued after the first r observations are made. The maximum likelihood estimate of θ is derived, $\hat{\theta}_{r,n}=\{x_{1,n}+x_{2,n}+\cdots+x_{r,n}+(n-r)x_{r,n}\}/r$, which is unbiased, efficient, sufficient, possesses minimum variance, and $(2r\hat{\theta}_{r,n})/\theta$ is distributed as χ^2 with $2r$ degrees of freedom. Let $E(X_{r,n})$ be the expected length of time needed to observe the first r out of n ordered observations. The ratio $E(X_{r,n})/E(X_{r,r})$, a measure of the expected saving in time, is tabulated for $n=1(1)5$, $10(5)20$ and $r=1(1)5$, 10 to two significant figures. The authors consider the test of the hypothesis H_0: $\theta=\theta_1$ against the alternative $\theta=\theta_2<\theta_1$. Methods are given for the determination of r based on the operating characteristic function $L(\theta)$ of the hypothesis under test. The values of r and the acceptance regions for fixed α, β, where α = probability of rejecting θ_1 when $\theta=\theta_1$, β = probability of accepting θ_1 when $\theta=\theta_2$ and where $\theta_1>\theta_2$ and the acceptance region is of the form $\hat{\theta}_{r,n}>C$ are tabled for $\theta_1/\theta_2=3/2$, 2, 5/2, 3, 4, 5, 10; α and β equal to .01, .05, and .10. Truncated and sequential tests are briefly discussed. The results are useful in life testing of equipment. *L. A. Aroian* (Culver City, Calif.).

Sobel, Milton. An essentially complete class of decision functions for certain standard sequential problems. Ann. Math. Statistics **24**, 319–337 (1953).

The author considers the problem of testing sequentially about a sequence of independently and identically distributed chance variables $\{X_i\}$ with common density $\psi(\theta)e^{\theta x}$ with respect to a σ-finite measure μ, the hypothesis that $\theta\leqq\theta^*$ against the alternative that $\theta>\theta^*$. Under rather general assumptions about the loss and cost functions, it is shown that an essentially complete class relative to the class of decision functions with bounded risk is given by stopping rules which for some sequences $\{b_n\leqq a_n\}$ of real numbers, terminate experimentation the first time the inequality $b_n<\sum_{i=1}^{n}X_i<a_n$ is violated (with possible randomization at the boundaries and the obvious terminal decision on stopping). The author's method is to reduce the problem to that of obtaining this structure for Bayes solutions which terminate, for some N, after at most N observations, and then to solve the latter problem by an induction on the stage of experimentation from N backwards. This last calculation, which is not easy, utilizes the monotone structure of the likelihood ratio for this family, a tool also exploited by the author in a similar treatment of the k-decision fixed sample-size problem [Thesis, Columbia Univ., 1951; this last is generalized in a forthcoming paper of Rubin]. *J. Kiefer* (Ithaca, N. Y.).

Konijn, H. S. A remark on the characterization of minimax procedures. Ann. Inst. Statist. Math., Tokyo **4**, 103–105 (1953).

A tool used in minimax proofs by Wald [Statistical decision functions, Wiley, New York, 1950, p. 167; these Rev. **12**, 193] and others is that δ is minimax if it is Bayes in the wide sense relative to some sequence $\{\xi_i\}$ and if $r(\xi_i,\delta)\to\sup_F r(F,\delta)$. The author proves a slightly weaker result (Theorem 2) and also (Theorem 1), under Wald's Assumptions 3.1–3.6, what is essentially the converse to the above. *J. Kiefer* (Ithaca, N. Y.).

Kawata, Tatsuo. On Wiener's prediction theory. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **2**, no. 4, 120–129 (1953).

Expository paper. *J. L. Doob* (Urbana, Ill.).

Whittle, P. The analysis of multiple stationary time series. J. Roy. Statist. Soc. Ser. B. **15**, 125–139 (1953).

In previous papers [Thesis, Uppsala, 1951; Biometrika **39**, 309–318 (1952); these Rev. **12**, 726; **14**, 666] the author has applied least square methods to estimation and testing problems for stationary time series. In the present paper he uses these methods in the multi-dimensional case. The emphasis is on practical formal solutions rather than rigorous theory. *J. L. Doob* (Urbana, Ill.).

TOPOLOGY

Ringel, Gerhard. Farbensatz für nichtorientierbare Flächen beliebigen Geschlechtes. J. Reine Angew. Math. **190**, 129–147 (1952).

Heawood proved that, for $h\geqq 2$, the number of colours which are necessary to colour the countries of any map, drawn on a surface of connectivity h, in such a way that any two countries which have a common frontier receive different colours, is at most $[3\frac{1}{2}+\frac{1}{2}(24h-23)^{1/2}]$. The subject matter of this paper is the colouring of maps on one-sided surfaces of arbitrary genus g ($g=h-1$). Let ν_g be such that a map containing ν_g mutually adjacent countries can be drawn on a one-sided surface of genus g but no map containing more than ν_g mutually adjacent countries can be drawn on any such surface; and let χ_g be such that a map which cannot be coloured with less than χ_g colours can be drawn on some one-sided surface of genus g, but no map which cannot be coloured with less than χ_g+1 colours can be drawn on any such surface. It follows from Heawood's colour formula that $\nu_g\leqq\chi_g\leqq[3\frac{1}{2}+\frac{1}{2}(24g+1)^{1/2}]$.

Let the right-hand member of the above inequality be denoted by H_g. It is known that for $g=2$, $\nu_g=\chi_g=H_g-1$. The author conjectures that $\nu_g=\chi_g=H_g$ for all $g\neq 2$, but he proves this only for $g=1$, for $3\leqq g\leqq 15$, and for all values of g such that $H_g\equiv 3 \pmod 6$. For all other values of g he proves that $H_g-2\leqq\nu_g=\chi_g\leqq H_g$. It is also shown that $\nu_{g+r}\geqq\nu_g+1$ for $r\geqq[\frac{1}{2}(\nu_g-2)]$. Suitable schemes are constructed which can represent maps in which any two countries have a common frontier, and these maps are formed into surfaces which always turn out to be one-sided.

It follows from the inequality $H_g-2\leqq\nu_g=\chi_g\leqq H_g$ that $\chi_g=H_g$ if and only if a complete H_g-graph can be drawn without crossing of edges on a one-sided surface of genus g, $\chi_g=H_g-1$ if no complete H_g-graph can be drawn on any one-sided surface of genus g but a complete (H_g-1)-graph

can be drawn on some such surface, and if neither a complete H_q-graph nor a complete (H_q-1)-graph can be drawn on any such surface, then $\chi_q=\nu_q=H_q-2$. It has of course often been conjectured that Heawood's upper bound, $[3\frac{1}{2}+\frac{1}{2}(24h-23)^{1/2}]$, is best possible, but this was proved only for $2\leqq h\leqq 16$ and for values of h with certain very special properties. This paper makes a very considerable contribution towards the solution of this problem for one-sided surfaces. *G. A. Dirac* (London).

Dirac, G. A. The colouring of maps. J. London Math. Soc. **28**, 476–480 (1953).

The chromatic number of a graph G is the least integer k such that the nodes of G can be coloured in k colours so that no two of the same colour are joined by an edge. For a surface S the author defines $k(S)$ as the greatest possible chromatic number for a graph on the surface. For an integer h he defines $k(h)$ as the greatest possible $k(S)$ for surfaces of connectivity h. It is known that

$$k(h)\leqq H(h)=[3\tfrac{1}{2}+\tfrac{1}{2}(24h-23)^{\frac{1}{2}}].$$

The author proves that if any graph of given chromatic number $k\geqq 8$ can be drawn on a given surface S then such a graph having at most $6(h-3)/(k-7)$ nodes can be drawn on S. Since $k(S)\leqq H(h)$ it follows that by testing a finite number of graphs we can either determine $k(S)$ or demonstrate that $k(S)\leqq 7$. It follows likewise that $k(h)$ can be determined by a finite process if it is known to exceed 7 (as is the case when $h>4$). *W. T. Tutte.*

Michael, Ernest. A note on paracompact spaces. Proc. Amer. Math. Soc. **4**, 831–838 (1953).

L'auteur démontre plusieurs résultats nouveaux et intéressants sur les espaces paracompacts. Un recouvrement d'un espace topologique est dit σ-localement fini s'il est réunion d'une infinité dénombrable de recouvrements localement finis. L'auteur prouve d'abord que la paracompacité, pour un espace régulier E, équivaut à la condition, en apparence moins forte, que tout recouvrement admet un recouvrement ouvert plus fin et σ-localement fini. Ceci entraîne en particulier, pour les espaces réguliers, l'identité de la notion d'espace paracompact et de celle d'espace "strongly screenable" de R. H. Bing [Canadian J. Math. **3**, 175–186 (1951); ces Rev. **13**, 264]. Un autre conséquence est que tout sous-espace d'un espace paracompact qui est un F_σ est paracompact. Enfin, l'auteur montre qu'un produit $X\times Y$ est paracompact dans les deux cas suivants: 1) X est paracompact, Y un espace régulier, réunion dénombrable de sous-espaces compacts [Note du rapporteur: cela entraîne donc que $X\times Y$ est paracompact lorsque X est paracompact, Y localement compact et paracompact]; 2) X est paracompact et parfaitement normal, Y est métrisable. *J. Dieudonné* (Evanston, Ill.).

Smirnov, Yu. On the completeness of proximity spaces. Doklady Akad. Nauk SSSR (N.S.) **88**, 761–764 (1953). (Russian)

If P is a proximity space, then a collection ξ of subsets of P is called a c-system if, for every uniform δ-covering [cf. Smirnov, Mat. Sbornik N.S. **31**(73), 543–574 (1952); these Rev. **14**, 1107] γ of P, there exists $X\in\xi$ and $\Gamma\in\gamma$ with $X\subset\Gamma$. The completion of P is defined, essentially, as the space cP of all maximal centered $c\delta$-systems [for δ-systems cf. Smirnov, same Doklady (N.S.) **84**, 895–898 (1952); these Rev. **14**, 1107]; P is called complete if $cP=P$. If P, Q are proximity spaces, then a mapping f of P into Q is called a c-mapping if the image of any centred c-system is a c-system. A proximity space is called totally bounded if any uniform δ-covering contains a finite subcovering.

The author gives (without proofs) many interesting theorems concerning completion, c-mappings, and total boundedness of proximity spaces. Some characteristic results: (1) cP is the largest of all δ-extensions R of P (i.e., proximity spaces containing P as a dense proximity subspace) such that every uniform δ-covering of P admits of an extension to a uniform δ-covering of R; (2) P is complete if and only if every centred c-system of closed sets has a non-void intersection; (3) a mapping f of P into Q may be extended to a continuous mapping of cP into cQ if and only if f is a c-mapping; (4) each of the following properties is equivalent with the total boundedness of P: (a) cP is compact; (b) every real-valued c-function on P is bounded; (c) every pseudometric in P is bounded; (b) there exists only one uniformity compatible with the proximity structure of P. An example is given of a proximity space possessing no maximal (finest) uniformity. *M. Katětov* (Prague).

Wada, Junzo. Lattices of spaces. Osaka Math. J. **5**, 1–12 (1953).

Let E be any abstract set. Let $\mathfrak{L}_T$ be the set of all topologies defined on the set E. Given two topologies R_1 and R_2, defined respectively by the families of open sets $O(R_1)$ and $O(R_2)$, one writes $R_1\leqq R_2$ if $O(R_1)\subset O(R_2)$. Under this definition, $\mathfrak{L}_T$ is a complete lattice [see, e.g., the reviewer's paper, Duke Math. J. **10**, 309–333 (1943); these Rev. **5**, 46]. Given a closure operator $^-$ for all subsets of E such that $0^-=0$ and $(A\cup B)^-=A^-\cup B^-$ for all A, $B\subset E$, one obtains a generalized sort of topology, and the set $\mathfrak{L}$ of all such topologies forms a complete lattice under the ordering $R_1\leqq R_2$ if the identity mapping is a continuous map of R_2 onto R_1. The family of all generalized topologies on E such that $A\subset A^-$ for all $A\subset E$ is denoted by $\mathfrak{L}_A$. It too is a complete lattice. A number of theorems about these lattices are proved, of which the following are typical examples. (1) If E contains more than one point, then no proper sublattice of $\mathfrak{L}$ contains all compact topologies in $\mathfrak{L}$. (2) No proper sublattice contains all T_0-spaces in $\mathfrak{L}$. (3) Every T_1-space in $\mathfrak{L}_T$ is homeomorphic to the diagonal in the Cartesian product of two bicompact T_1-spaces of $\mathfrak{L}_T$. (4) Let S be an hereditary property of a space in $\mathfrak{L}_T$. If $\{R_\lambda\}_{\lambda\in\Lambda}\subset\mathfrak{L}_T$ is such that all R_λ enjoy the property S and if $P_{\lambda\in\Lambda}R_\lambda$ enjoys the property S, then $\sup_{\lambda\in\Lambda}R_\lambda$ also does. (5) A completely regular space in $\mathfrak{L}_T$ is minimal in the class of completely regular spaces in $\mathfrak{L}_T$ if and only if it is bicompact.
E. Hewitt (Seattle, Wash.).

Pierce, R. S. The Boolean algebra of regular open sets. Canadian J. Math. **5**, 95–100 (1953).

Let $C(S)$ denote the lattice of bounded continuous real-valued functions on a completely regular topological space S. For $f\in C(S)$, $g\in C(S)$ let $f\sim g$ if, for any $h\in C(S)$, $f\cap h\leqq 0$ if and only if $g\cap h\leqq 0$. Then $\sim$ is a congruence relation; let $\mathfrak{L}(S)$ denote the lattice obtained by reducing $C(S)$ modulo $\sim$. It is shown that $\mathfrak{L}(S)$ is isomorphic to a sublattice of the Boolean algebra $\mathfrak{B}(S)$ of all regular open sets of S. Main results of the note: $\mathfrak{B}(S)$ is isomorphic to the normal completion of $\mathfrak{L}(S)$ and also to the lattice obtained by reducing the normal completion of $C(S)$ [for the properties of the normal completion of $C(S)$ cf. R. P. Dilworth, Trans. Amer. Math. Soc. **68**, 427–438 (1950); these Rev. **11**, 647] modulo the congruence relation $\sim$ (extended, in an evident way, to the completion of $C(S)$). The author

gives, too, a direct topological characterization of $\mathfrak{L}(S)$ in the case of a normal space S. *M. Katětov* (Prague).

Nöbeling, Georg. Limitentheorie in topologischen Vereinen und Verbänden. J. Reine Angew. Math. **191**, 125–134 (1953).

The program of the paper is to generalize to topological partially ordered sets results concerning superior and inferior limits in topological spaces. A topological lattice is defined as a lattice with a closure operator satisfying the usual Kuratowski axioms. Suppose that I is any set and that a (downwards) directed class R of subsets of I is given. An element A of a topological lattice is adherent (or strictly adherent) along R to a family $\{A_i\}$ of elements, $i \in I$, if $A \leqq S$ whenever $A_i \leqq S$ for all i in some r of R (or for some i in all r of R). Typical theorem: if the topological lattice is distributive, and if an atom P is adherent to a family $\{A_i\}$ along a directed class R of subsets of I, then there exists a finer directed class R^* such that P is strictly adherent to $\{A_i\}$ along R^*. *P. R. Halmos* (Chicago, Ill.).

Katětov, M. On the dimension of metric spaces. Doklady Akad. Nauk SSSR (N.S.) **79**, 189–191 (1951). (Russian)

Let P be a topological space; let dim P be the dimension of P defined by finite coverings; and let Ind P be the "large" inductive dimension, defined by the boundaries of neighborhoods of closed sets. It is known that dim $P \leqq$ Ind P if P is normal [N. Vedenisov, Izvestiya Akad. Nauk SSSR. Ser. Mat. **5**, 211–216 (1941); these Rev. **3**, 58]. If P is a metric space with a countable basis, then, as is well-known, dim P = Ind P. It is announced here, and a proof is sketched, that dim P = Ind P for all metric spaces. Several other equivalent statements are proved. An interesting corollary is the fact that if P is a metric space, then dim $P \leqq 0$ if and only if P is homeomorphic to a subspace of a countable Cartesian product of discrete spaces. *E. Hewitt.*

Boltyanskiĭ, V. On the theorem of addition of dimensions. Uspehi Matem. Nauk (N.S.) **6**, no. 3(43), 99–128 (1951). (Russian)

This is a detailed exposition of two constructions published previously. The first is L. Pontryagin's construction of a sequence of 2-dimensional compact metric spaces $F_2, F_3, \cdots, F_m, \cdots$ such that dim $(F_m \times F_n) = 3$ if m and n are relatively prime [C. R. Acad. Sci. Paris **190**, 1105–1107 (1930)]. The second is a construction of the author, giving a class of 2-dimensional compact metric spaces P_m (m a prime) such that dim $(P_m \times P_m) = 3$ [Doklady Akad. Nauk SSSR (N.S.) **67**, 597–599 (1949); these Rev. **11**, 45]

E. Hewitt (Seattle, Wash.).

Boltyanskiĭ, V. G. A new geometric characteristic of the Urysohn dimension. Mat. Sbornik N.S. **29(71)**, 603–614 (1951). (Russian)

Let F be a compact subset of some finite-dimensional Euclidean space E. Let $\tau = \{a_0, a_1, \cdots, a_r\}$ be any subset of $r+1$ distinct points of F. Such a set τ is called an r-dimensional skeleton in F, and each point a_i is a vertex of τ. A complex γ in F is any collection of skeletons such that $\tau \in \gamma$ and $\tau' \subset \tau$ implies $\tau' \in \gamma$. The dimension of γ is the largest of the dimensions of $\tau \in \gamma$, and diam γ is defined as $\max_{\tau \in \gamma}$ diam τ. Let A be a closed proper subset of F. An r-dimensional complex γ in F is called a geometric cycle with respect to A if for every $(r-1)$-dimensional skeleton $\tau \in \gamma$ such that $\tau \not\subset A$, there exist at least two r-dimensional skeletons $\tau_1, \tau_2 \in \gamma$ such that $\tau_1 \supset \tau$, $\tau_2 \supset \tau$. Let γ be a complex and let $F^* = \{a_0, a_1, \cdots, a_s\}$ be the set of all vertices of skeletons in γ. Any mapping f of F^* into E defines a new complex, $f(\gamma)$, such that $\{f(a_{i_1}), \cdots, f(a_{i_p})\}$ is a skeleton in $f(\gamma)$ if and only if $\{a_{i_1}, \cdots, a_{i_p}\}$ is a skeleton in γ. Now, if f is a mapping of the entire compact set F into E, and if γ^r is a geometric cycle in F with respect to $A \subset F$, $A \neq F$, then $f(\gamma^r)$ may or may not be a geometric cycle in $f(F)$ with respect to $f(A)$. [Remark: It is necessary here to suppose that f is continuous.] If $f(\gamma^r)$ admits a subcomplex which is a non-trivial r-dimensional geometric cycle in $f(F)$ with respect to $f(A)$, then γ^r is said to be non-degenerate under f relative to A. If γ^r is non-degenerate under all mappings f of F for which dist $(x, f(x)) < \epsilon$ for all $x \in F$, then γ^r is said to be ϵ-essential with respect to A.

Consider a sequence $\Gamma^r = (\gamma_1^r, \gamma_2^r, \cdots, \gamma_i^r, \cdots)$ of r-dimensional geometric cycles γ_i^r with respect to a fixed $A \subset F$ such that $\lim_{i \to \infty}$ diam $\gamma_i^r = 0$. Suppose that there exists $\epsilon > 0$ such that for an infinite number of indices i, γ_i^r is ϵ-essential relative to A. The geometric dimension, $d_{\text{geom}}(F)$, is defined as the greatest natural number r for which there exists $A \subset F$ and Γ^r having the properties set forth above. It is proved that $d_{\text{geom}}(F)$ is equal to the Urysohn dimension ind F. Several examples are constructed, and a final remark asserts that the definition of $d_{\text{geom}}F$ and the theorem enunciated can be extended easily to arbitrary compact metric spaces, without regard to imbedding in E.

E. Hewitt (Seattle, Wash.).

O'Neill, Barrett. Essential sets and fixed points. Amer. J. Math. **75**, 497–509 (1953).

Let X be a topological space, X^X the space of continuous mappings of X into itself under the compact-open topology. The author calls a subset S of X "essential" with respect to $f \in X^X$ if given any neighborhood $V(S)$ there is a neighborhood $N(f)$ such that each member of $N(f)$ has a fixed point in $V(S)$. This generalizes the essential fixed points of M. K. Fort [same J. **72**, 315–322 (1950); these Rev. **11**, 609]. If U is an open set in X with no fixed points of f on its boundary, an integer-valued trace function $L(f, U)$ is defined which has the properties of a fixed point index; $L(f, U)$ serves to recognize and locate essential components of the set of fixed points of f. It is proved that L is the only topologically invariant fixed point index on the class of finite polyhedra. *S. B. Myers* (Ann Arbor, Mich.).

***Kuratowski, Casimir. Sur une propriété topologique fondamentale du plan euclidien.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 361–362. Casa Editrice Perrella, Roma, 1953.

This is a summary of Grzegorczyk and Kuratowski, Ann. Soc. Polon. Math. **25**, 169–182 (1953); these Rev. **14**, 1108.

A. H. Stone (Manchester).

***Denjoy, Arnaud. A propos des théorèmes dits "de Janiszewski" (communication consecutive à un exposé de M. C. Kuratowski).** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 363–365. Casa Editrice Perrella, Roma, 1953.

In connection with a paper by Kuratowski [see the preceding review], the author points out that in 1911 [C. R. Acad. Sci. Paris **153**, 493–496 (1911)] he had announced some theorems which are related to results of Janiszewski [Prace Mat.-Fiz. **26**, 11–63 (1915)], but which Janiszewski had apparently overlooked. Here Denjoy restates his 1911 results, and gives the following generalization of them: If

C_1, C_2 are plane continua such that $C_{2-i}-C_i$ is in a single component R_i of $C_0(C_i)$ $(i=1, 2)$, and if $C_1 \cap C_2$ has p components (where p is any positive integer), then $R_1 \cap R_2$ has p components also. [It seems to the reviewer, however, that this is more closely related to work of Straszewicz [Fund. Math. 7, 159–187 (1925)], from which it can be deduced, than it is to "Janiszewski's theorem".]

A. H. Stone (Manchester).

Kapuano, Isaac. Sur une proposition de M. Bing. C. R. Acad. Sci. Paris **236**, 2468–2469 (1953).

The following question was raised by Knaster and Kuratowski [Fund. Math. 1, 223 (1920), Problème 2]: If a nondegenerate bounded plane continuum is homogeneous, is it necessarily a simple closed curve? Mazurkiewicz [ibid. 5, 137–146 (1924)] gave a partial solution to the problem by showing that the simple closed curve is the only homogeneous nondegenerate bounded locally connected plane continuum. Waraszkiewicz [C. R. Acad. Sci. Paris **204**, 1388–1390 (1937)] announced that the answer to the original question was in the affirmative. Choquet [ibid. **219**, 542–544 (1944); these Rev. 7, 335] announced a classification of all homogeneous bounded closed plane sets. R. H. Bing in an extensive paper [Duke Math. J. **15**, 729–742 (1948); these Rev. 10, 261] gave, among other things, an example of a homogeneous nondegenerate bounded plane continuum which is not a simple closed curve.

The present author defines what he calls an extremity of a continuum, and presents an argument for the result that a bounded homogeneous plane continuum can have no extremity. He concludes from this result that the argument given by Bing to prove the continuum of his example homogeneous does not apply. *D. W. Hall.*

Vaughan, H. E. On two theorems of plane topology. Amer. Math. Monthly **60**, 462–468 (1953).

This paper (essentially pedagogical) contains a proof of: If C is a continuum in the plane and G is a region whose complement is contained in the outer region determined by C, then there is a simply connected subregion of G which contains C. The result is used in complex analysis.

A. D. Wallace (New Orleans, La.).

Saalfrank, C. W. On the universal covering space and the fundamental group. Proc. Amer. Math. Soc. **4**, 650–653 (1953).

Let A be a retract of X. The author gives detailed proofs of the following known facts. (1) The universal covering space of A is homeomorphic to a retract of the universal covering of X. (2) The fundamental group of A is isomorphic to a retract of the fundamental group of X.

J. Dugundji (Los Angeles, Calif.).

Ganea, Tudor. Covering spaces and cartesian products. Ann. Soc. Polon. Math. **25** (1952), 30–42 (1953).

A cartesian product E has a non-trivial covering if and only if some one of its factors E_λ has a non-trivial covering. In order that E have a universal covering $\bar{E}$ it is necessary and sufficient that each factor E_λ have a universal covering $\bar{E}_\lambda$ and that $\bar{E}_\lambda$ is almost always E_λ. Necessary and sufficient for the unicoherence of E is the unicoherence of each E_λ.

R. H. Fox (Princeton, N. J.).

Hopf, H. Vom Bolzanoschen Nullstellensatz zur algebraischen Homotopietheorie der Sphären. Jber. Deutsch. Math. Verein. **56**, 59–76 (1953).

This paper is based on an expository lecture given by the author in September, 1951. The main purpose is to trace the development of the problem of determining the homotopy groups of spheres down to the present. The starting point is the old theorem of Bolzano to the effect that if f is a continuous real-valued function defined on the closed interval $a \leqq x \leqq b$, and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x)=0$ has a solution. By attempting to generalize this theorem to higher dimensions, the author shows how one is led to consider the following two problems. (I) Given a continuous map of one sphere onto another, how can one determine whether or not it is homotopic to a constant? (II) Given two integers m and n, does there exist an essential map of an m-sphere onto an n-sphere? Various developments along these lines such as the Brouwer degree, the Kronecker integral, and Hopf invariant are next taken up, leading eventually to consideration of the Hurewicz homotopy groups. The author then outlines in historical order the various results which have been obtained about homotopy groups of spheres, including many of the recent results of J. P. Serre and H. Cartan. *W. S. Massey.*

Hu, Sze-tsen. The homotopy addition theorem. Ann. of Math. (2) **58**, 108–122 (1953).

The author gives a detailed, and elementary, proof of the homotopy addition theorem. The main feature consists in using the basic n-simplex (instead of the n-sphere) as antiimage in defining the homotopy groups. *J. Dugundji.*

Morse, Marston, and Baiada, Emilio. Homotopy and homology related to the Schoenflies problem. Ann. of Math. (2) **58**, 142–165 (1953).

The first part of this paper is concerned with the topological relations between a smoothly imbedded compact n-manifold $M_n E_{n+1}$, and the closure of its interior IM_n.

The principal tool of the investigation is the study of the critical points of functions defined on M_n by virtue of the imbedding of M_n in E_{n+1}. Thus the authors start by showing that given $M_n \subset E_{n+1}$ and any direction v in E_{n+1}, there is a Euclidean coordinate system $(x_1, \cdots, x_{n+1})$ in E_{n+1} such that the positive direction of the x_{n+1} axis is arbitrarily close to v, and $f=x_{n+1}|M_n$ is nondegenerate (i.e., the critical sets of f are points which occur at different f levels and the Hessian of f at each critical point is non-degenerate).

The critical points of f are classified into the sup and inf critical points, depending on whether the outgoing normal of M_n at the critical point is in the positive direction of the x_{n+1} axis or not. The relations of f on M_n and x_{n+1} on IM_n are studied by means of "inf trajectories". These are simple arcs, which, proceeding from an inf critical point of f on M_n pass through $IM_n - M_n$ and end on M_n in such a fashion that (1) every point of M_n except for the inf critical points of f is a terminal point of some trajectory, (2) there passes at least one inf trajectory through every point Q of IM_n and only one if Q is not inf critical, and (3) the following continuity condition is satisfied: if $P_r \to P_0$ where all P_r and P_0 are inf ordinary, and if $z_r \to z_0$ is any sequence of values for which $g^1(z_r)$ (g^r the unique trajectory through P_r) is defined and $g^0(z_0)$ is inf ordinary, then $g^r(z_r) \to g^0(z_0)$.

As a sample of the theorems which follow quite readily from the existence of inf trajectories we mention the following proposition: if f is nondegenerate and has but two critical

points at Y_1 and Y_2 at which $f=1$ and -1 respectively, then there exists a homeomorphism of IS^n (S^n is the usual unit sphere) onto IM_n in which points in IM_n correspond to points in IS_n at the same x_{n+1} level.

The authors go on to describe the following model for IM_n. Let the critical values of $f=x_{n+1}|M_n$ be written in ascending order $c_0<c_1\cdots<c_m$. In each interval $c_i\leqq x\leqq c_{i+1}$ for which the critical points corresponding to c_i and c_{i+1} are of different type, choose a value a_i. For every such "separating value" a_i consider the "prism" $\Pi(a_\tau)$ defined by $\Pi(a_\tau)=$union $(x_1,\cdots,x_n,a_\tau+t)\subset E_{n+1}$, $(x_1,\cdots,x)\in x_{n+1}^{-1}(a_\tau)$, $t\in[-d,d]$, where d is a positive number so small that there are no critical values of x_{n+1} in any of the intervals $a_\tau-d\leqq x_{n+1}\leqq a_\tau+d$. The model for IM_n is obtained by identifying in the union of the disjoint prism $\Pi(a_\tau)$ certain sets of the upper base of $\Pi(a_\tau)$ with certain sets in the lower base of $\Pi(a_{\tau+1})$. With the aid of this model many "inessential" critical points of f are avoided. For instance, it follows as an easy consequence that if all the critical points of $x_{n+1}|M$ are inferior (superior) except for the absolute maximum (minimum) of f, then IM_n is a topological $(n+1)$-cell.

The authors then apply this decomposition theorem in many ways to problems of the Schoenflies type. The second part of the paper is devoted to deformation problems of submanifolds on manifolds. We quote the last theorem of the paper: Let F be of class C^3 and nondegenerate on a 5-regular M_n in E_p, $p>n$. Let $[a,b]$ be an interval of values of F which contains just one critical value $c\in(a,b)$ taken on by a critical point P_0 of index $k>1$. Let r be an integer such that $0<r<k$. Then any 2-regular manifold $M_r\subset M_n$ below b can be isotopically deformed through 1-regular manifolds into a 2-regular manifold below a. *R. Bott.*

Plans, Antonio. Contribution to the study of the homology groups of the cyclic ramified coverings corresponding to a knot. Revista Acad. Ci. Madrid **47**, 161–193 (5 plates) (1953). (Spanish)

Seifert [Math. Ann. **110**, 571–592 (1934)] has shown that the first homology group H_g of the gth cyclic covering of a knot of genus $\leqq h$ is determined by a relation matrix of the form $F_g=\Gamma^g-(\Gamma-E)^g$, where Γ is a certain matrix of order $2h$ that satisfies the condition (*) $\Gamma'=E+I\Gamma I$, where $I=\sum_{j=1}^{h}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. The author derives from this, by algebraic manipulations, various conclusions about the structure of H_g. The most striking result is that H_{2n+1} is always the direct sum of some group with itself. This fact has surely been conjectured by everyone who has examined the knot table [for example in Reidemeister, Knotentheorie, Springer, Berlin, 1932, p. 25] and observed that for $g=3$ the torsion number occurs in pairs. The complete statement is that

$$F_g\sim\sum_{j=1}^{h}\begin{pmatrix}a_j&0\\0&a_j\end{pmatrix}\quad\text{for odd } g,$$

$$\sim F_2\cdot\sum_{j=1}^{h}\begin{pmatrix}b_j&0\\0&b_j\end{pmatrix}\quad\text{for even } g.$$

The author's proof is a somewhat lumbering reduction to diagonal form, that could have been deduced much more simply from the classical reduction of a skew-symmetric form to its block-diagonal normal form by observing that $F_{2n+1}\cdot I$ and $F_{2n}\cdot F_2^{-1}\cdot I$ are skew symmetric matrices. (This follows easily from (*).)

It is shown that

$$(**)\qquad F_g=(2\Gamma-E)\cdot P^{(g)}\quad\text{for even } g,$$
$$=P^{(g)}-2TP^{(g-1)}\quad\text{for odd } g,$$

where $T=\Gamma(E-\Gamma)$ and $P^{(g)}$ is defined recursively by $P^{(0)}=0$, $P^{(1)}=E$, $P^{(g+1)}=P^{(g)}-T\cdot P^{(g-1)}$. These expressions are of importance in determining more precisely the structure of H_g.

For knots of genus $h=1$, remarkable simplification follows from the fact that T is then always diagonal. The formula (**) becomes

$$(***)\qquad F_g=p_g(2\Gamma-E)\quad\text{for even } g,$$
$$=(p_g-2\gamma p_{g-1})E\quad\text{for odd } g,$$

where $\gamma=\det\Gamma$ and p_g is the integer defined recursively by $p_0=0$, $p_1=1$, $p_{g+1}=p_g-\gamma p_{g-1}$. The explicit nature of (***) allows the following deductions to be made. (1) For knots of genus 1, all of the groups H_g are determined by γ, hence by H_2. (This was first noted by Seifert [Abh. Math. Sem. Hamburg. Univ. **11**, 84–101 (1935)].) (2) H_g is a finite group for all values of γ and g except $\gamma=1$ and $g\equiv0\pmod 6$; H_g is then free abelian of rank 2. (3) If $\gamma=0$, $H_g=0$ for every g; if $\gamma=1$, $H_{g+6}=H_g$; these are the only values of γ for which H_g is periodic in g.

The reviewer was unable to follow the arguments in several places; however the stated results are all correct. This paper complements results announced by the reviewer [Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 453–457; these Rev. **13**, 966]. *R. H. Fox.*

Fáry, István. Sur une nouvelle démonstration de l'unicité de l'algèbre de cohomologie à supports compacts d'un espace localement compact. C. R. Acad. Sci. Paris **237**, 552–554 (1953).

This note sketches what seems to be a simple and elegant proof of the uniqueness of the cohomology algebra of a locally compact space with compact carriers. We quote the author's own summary: "Soit $H(X,A)$ l'algèbre de cohomologie à supports compacts de l'espace X à coefficients dans A. Désignons par $\mathrm{kat}(h)$, $h\in H(X,A)$, le plus petit entier m, tel qu'il existe m fermés $F_1,\cdots,F_m$ recouvrant X, et tels que h soit annulé par les homomorphismes naturels $H(X,A)\to H(F_i,A)$, pour $i=1,\cdots,m$. Dans le cadre de la théorie de J. Leray cette fonction peut servir de paramètre d'induction dans la démonstration de l'unicité de $H(X,A)$."

S. Chern (Chicago, Ill.).

Yoneda, Nobuo. On the mappings of complexes. J. Fac. Sci. Univ. Tokyo. Sect. I. **6**, 393–419 (1953).

Let K, K' be two simplicial subdivisions of a finite polytope P having a common stellar subdivision. The author shows that there exists a finite sequence of simplicial subdivisions $K=K_0,\cdots,K_n=K'$ of P and a finite sequence $h_i(p,t)$, $0\leqq t\leqq1$, of deformations of P such that each $h_i(p,1)$ is an isomorphism of a subdivision of K_{i+1} onto a subdivision of K_i. He also shows that every continuous map of a finite polytope P into a combinatorial manifold M can be deformed, with arbitrarily small deformation, into a simplicial map f of a simplicial subdivision of P for which $\max_{\sigma\in P}[\dim\sigma-\dim f(\sigma)]=\max[\dim P-\dim M,0]$. These results are applied to define, and give invariance proofs for an "absolute degree" of a map of one manifold into another, which is a special case of Olum's "twisted degree" [Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. II, Amer. Math. Soc., Providence, R. I., 1952, pp. 363–370; these Rev. **13**, 485]. *J. Dugundji* (Los Angeles, Calif.).

Hawley, N. S. A note on projective line bundles. Ann. of Math. (2) **58**, 366–370 (1953).

This paper may be regarded as a preliminary to an attack on the problem of analyzing regular projective bundles into equivalence classes [Hawley, Proc. Nat. Acad. Sci. U. S. A. **38**, 411–415 (1952); these Rev. **14**, 73]. It partially motivates this problem by presenting several methods of generating projective line bundles, and pointing out that all these bundles are equivalent to regular projective bundles.

S. S. Cairns (Urbana, Ill.).

GEOMETRY

Gerretsen, J. C. H. Inequalities in the triangle. Nieuw Tijdschr. Wiskunde **41**, 1–7 (1953). (Dutch)

Considering a triangle of sides a, b, c, semiperimeter s, inradius r, circumradius R, area Δ, and "scalenity"

$$Q=(b-c)^2+(c-a)^2+(a-b)^2,$$

the author gives simple proofs for the inequalities

$$a^2+b^2+c^2\geqq 4\Delta\sqrt{3},\quad a^2+b^2+c^2\geqq 4\Delta\sqrt{3}+Q$$

of Weitzenböck [Math. Z. **5**, 137–146 (1919)] and Finsler and Hadwiger [Comment. Math. Helv. **10**, 316–326 (1938)]. The latter is clearly equivalent to

$$\Delta\sqrt{3}\leqq s^2-\tfrac{1}{2}(a^2+b^2+c^2).$$

The following new results are deduced:

$$\Delta\sqrt{3}\leqq 4Rr+r^2\leqq\Delta\sqrt{3}+\tfrac{1}{2}Q,$$
$$8Rr\leqq s^2-11\Delta/3\sqrt{3}\leqq 4R^2.$$

In each case, equality holds only when $a=b=c$.

H. S. M. Coxeter (Toronto, Ont.).

Locher-Ernst, L. Bilder zur Geometrie der regelmässigen Figuren. Elemente der Math. **8**, 97–102 (1953).

Singh, Kuldip. Applications of the eleven point conic. Amer. Math. Monthly **60**, 468–474 (1953).

Nisnevič, L. B., and Bryzgalov, V. I. On a problem of n-dimensional geometry. Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 169–172 (1953). (Russian)

In an n-dimensional Euclidian space there are given n mutually orthogonal vectors $a_1, \cdots, a_n$, with respective lengths $\alpha_1, \cdots, \alpha_n$. For a given m, $m\leqq n$, what is the condition that there exists an m-dimensional subspace L_m in which the vectors $a_1, \cdots, a_n$ have projections of equal length? The authors answer with the theorem: The necessary and sufficient condition that L_m exists is that $\alpha_i^2(\alpha_1^{-2}+\cdots+\alpha_n^{-2})\geqq m$ $(i=1, 2, \cdots, n)$. *W. E. Milne.*

Charrueau, André. Sur certains systèmes de transformations. C. R. Acad. Sci. Paris **236**, 1529–1531 (1953).

Certain pencils of polarities relative to linear complexes and pencils of inversions satisfy: 1) $T'=TT_1T_2$ belongs to the pencil if T, T_1 and T_2 do; and 2) the parameter of T' is related to that of T by a homography with double points which do not depend on T_1 and $T_2\neq T_1$. *W. Givens.*

Bernays, Paul. Über die Verwendung der Polygoninhalte an Stelle eines Spiegelungsaxioms in der Axiomatik der Planimetrie. Elemente der Math. **8**, 102–107 (1953).

The author shows how a satisfactory geometry of the Euclidean plane, without continuity, can be deduced from the following seven sets of axioms. (1) Axioms of incidence. (2) Axioms of order. (3) Axioms of congruence for segments. (4) Axioms of congruence for angles. (5) An axiom of congruence for triangles of like sense. (6) An axiom of parallelism somewhat weaker than Euclid's. (7a) The areas of triangles are a system of magnitudes. (7b) Congruent triangles of like sense have equal area. (7c) Area is additive for dissection of a triangle into two triangles.

For the theory of proportion, the ratio of two segments is represented by the angle that has this ratio for its tangent. Some further simplification might have been effected had Hilbert's axioms of congruence been replaced by R. L. Moore's [cf. H. G. Forder, The foundations of Euclidean geometry, Cambridge, 1927, pp. 91–111].

H. S. M. Coxeter (Toronto, Ont.).

Morduhaĭ-Boltovskoĭ, D. D. A three- and a four-dimensional analogue of Pascal's theorem. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 135–138 (1953). (Russian)

The following theorem is proved. Let T be a tetrahedron and Q a quadric in P^3. For any triple of points on Q which lies on three edges of T through the same vertex of T form the intersection of the plane through the triple with the face of T opposite to the vertex. All lines obtainable in this way are rulings of the same hyperboloid. The theorem is dualized and extended to P^4. *H. Busemann.*

Mokriščev, K. K. On solvability of constructive problems of second degree in the Lobačevskiĭ plane by means of hypercycle or compasses and horocycle. Doklady Akad. Nauk SSSR (N.S.) **91**, 453–456 (1953). (Russian)

The title describes the content of the paper completely.

H. Busemann (Los Angeles, Calif.).

***Hirsch, G. A connection between projective geometry and some problems of topology.** Zeven voordrachten over topologie. [Seven lectures on topology]. Centrumreeks, no. 1. Math. Centrum Amsterdam, pp. 85–102. J. Noorduijn en Zoon, Gorinchem, 1950. 6 florins. (Dutch)

Szász, Paul. Beweis der Hauptformel der hyperbolischen Trigonometrie unabhängig von der Stetigkeit. Acta Sci. Math. Szeged **15**, 57–60 (1953).

Let s and s' (with $s<s'$) be corresponding arcs of two concentric horocycles with radial separation a [see, e.g., H. S. Carslaw, The elements of non-Euclidean plane geometry and trigonometry, Longmans Green, London, 1916, pp. 95, 109]. Then $s'/s=e^a=\cot\frac{1}{2}\alpha$, where $\alpha=\Pi(a)$. The author gives a new derivation of this relation between s'/s and α, avoiding considerations of continuity in the manner of Hilbert [Math. Ann. **57**, 137–150 (1903), pp. 139–140] and Engel [Nikolaj Iwanowitsch Lobatschefskij, Teubner, Leipzig, 1898, pp. 12–21]. *H. S. M. Coxeter.*

***Rozenfel'd, B. A. Non-Euclidean geometries over the complex and the hypercomplex numbers and their application to real geometries.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951. [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 151–166. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

Instead of assigning real values to the coordinates $x^0, x^1, \cdots, x^n$ of a point of a projective space P_n, we can

take them as complex, dual, quaternionic or pseudo-quaternionic numbers: 1) $a+bi$; 2) $a+be$; 3) $a+bi+cj+dk$; 4) $a+bi+ce+df$, where $i^2=j^2=-1$, $ij=-ji=k$, $e^2=i$, $ie=-ei=f$; the units of the dual numbers can also be taken as $e_+=\frac{1}{2}(1-e)$, $e_-=\frac{1}{2}(1-e)$, $e_+^2=e_+$, $e_-^2=e_-$, $e_+e_-=0$. Such spaces are accordingly denoted by $P_n(i)$, $P_n(e)$, $P_n(i, j)$, and $P_n(i, e)$. We can consider affine spaces A_n and unitary non-euclidean spaces K_n in a similar manner. A number of correspondences can now be established. For instance, the point of $P_n(e)$ given by $x^k=X^ke_++Y^ke_-$ represents a pair of points X^k and Y^k of a real P_n; if we pass to kX^k and lY^k the x^k are multiplied by (ke_++le_-). Moreover, lines, planes and hyperplanes of $P_n(e)$ represent pairs of such figures in P_n. The points of a $P_n(i, e)$ represent lines of a real P_{2n+1}. Since quaternions and pseudoquaternions do not satisfy the commutative law, the theorem of Pappus does not hold in $P_n(i, j)$ and $P_n(i, e)$. The metric of $K_n(i)$ and $K_n(i, j)$, if considered as real spaces of twice and four times their number of dimensions, is Riemannian. They belong to the symmetrical spaces of E. Cartan; $K_n(i)$ is a special A-space of P. A. Širokov [see A. P. Širokov, this same book, pp. 195–200; these Rev. 15, 62] and $K_n(e)$ a special stratifiable space of P. K. Raševskiĭ [Trudy Sem. Vektor. Tenzor. Analizu 6, 225–248 (1948); these Rev. 15, 62]. The $K_n(i)$ were first studied by Fubini and Study. They are isometric with a paratactic line congruence in a real Riemannian S_{2n+1} (with proper definition of metric). The $K_n(e)$, $K_n(i, j)$, $K_n(i, e)$ are also isometric with such a congruence, now in $a^{n+1}S_{2n+1}$ (pseudo-Riemannian with signature n, $n+1$), a $K_{2n+1}(i)$ and $K_{2n+1}(e)$ respectively. The $K_n(e)$ can also be made isometric with the manifold of the configurations (point-hyperplane) in P_n (with proper definition of metric). For the $K_n(i, e)$ this leads to a symplectic space Sp_{2n+1} which is the space of projective transformations in P_{2n+1} which leaves a skew-symmetric form

$$X^0Y^{n+1}-X^{n+1}Y^0+\cdots+X^0Y^{2n+1}-X^{2n+1}Y^0$$

invariant. *D. J. Struik* (Cambridge, Mass.).

Barthel, Woldemar. Zum Inhaltsbegriff in der Minkowskischen Geometrie. Math. Z. 58, 358–375 (1953).

The results of a paper by the reviewer [Comment. Math. Helv. 24, 156–187 (1950); these Rev. 12, 527] are treated again with a twofold purpose: (1) to give a purely Minkowskian treatment, i.e., to avoid euclidean arguments; (2) to create a formalism which permits access through calculation. The paper is only partially successful regarding the first objective inasmuch as for the more difficult results reference is made to the reviewer's paper which does use euclidean methods. It is highly successful with respect to the second objective. Let $x_1, \cdots, x_p$ be vectors in an n-dimensional Minkowski space with the same origin and denote by $F^{(p)}(x_i)=F^{(p)}(x_1, \cdots, x_p)$ the Minkowski volume of the parallelepiped spanned by $x_1, \cdots, x_p$. The properties of $F^{(p)}(x_i)$ are studied in detail and it is shown that it has many properties analogous to $F^{(1)}(x)$, like homogeneity, differentiability, etc. The reviewer's sine function, curvature, and other concepts are expressed very elegantly in terms of the $F^{(p)}(x_i)$. The curvatures of curves on a hypersurface are studied.

It is assumed that $F^{(1)}(x)$ is of class C''' and that $F^{(1)}(x)=1$ is strictly convex. This facilitates the calculations, but precludes solvability of many problems which arise naturally: for instance, in two dimensions the extrema of the α-function are attained by geometries whose unit circles are polygons. *H. Busemann* (Los Angeles, Calif.).

Buckel, Walter. Eine Kennzeichnung des Systems aller Kreise mit nichtverschwindendem Radius der euklidischen Ebene. J. Reine Angew. Math. 191, 13–29 (1953).

Let K denote the set of all circles of the euclidean plane E_2, with positive (or infinite) radii. As a class of subsets of E_2, the set K has the following properties: (1) K contains at least two distinct elements; (2) every element of K contains at least two distinct points, and any two distinct points of an element are joined by a simple arc that lies wholly in that element; (3) any three pairwise distinct points of E_2 are contained in exactly one element of K; and (4) each motion of the plane maps K onto itself. The paper proves that these four obvious properties of the set K characterize that set among the class of all systems of planar pointsets. The proof, based on elementary considerations alone, is quite lengthy. *L. M. Blumenthal* (Columbia, Mo.).

Buckel, Walter. Eine Kennzeichnung des Systems aller nichtzerfallenden Kegelschnitte der projektiven Ebene. J. Reine Angew. Math. 191, 165–178 (1953).

Let S denote the set of all non-degenerate conics of the projective plane P_2. As a system of pointsets of P_2, the set S has the following properties: (1) S contains at least two distinct elements; (2) every element of S contains at least three non-collinear points, and any two points of an element of S are joined by a simple arc that lies wholly in that element; (3) each five pairwise distinct points of P_2, no three of which are collinear, are contained in at least one element of S, while no quintuple of pairwise distinct points of P_2 is contained in more than one element of S; (4) each projective transformation of P_2 sends S into itself. The paper establishes that S is the only class of subsets of P_2 that has these four properties. Use is made of the results proved by the author in the paper reviewed above. *L. M. Blumenthal* (Columbia, Mo.).

Gupta, Hansraj. Non-concyclic sets of points. Proc. Nat. Inst. Sci. India 19, 315–316 (1953).

This note proves (by complete induction) that if a finite number of points in a plane are non-concyclic, and a circle is drawn through each three of them, then at least one of the circles so obtained contains exactly three points. The author seems unaware of a paper by Motzkin [Trans. Amer. Math. Soc. 70, 451–464 (1951); these Rev. 12, 849] in which this and other more general variants of the Sylvester-Gallai theorem are established. *L. M. Blumenthal.*

Convex Domains, Extremal Problems, Integral Geometry

Hammer, Preston C., and Sobczyk, Andrew. Planar line families. II. Proc. Amer. Math. Soc. 4, 341–349 (1953).

Suite de l'étude de même titre [Proc. Amer. Math. Soc. 4, 226–233 (1953); ces Rev. 14, 787]. Les auteurs donnent ici des conditions analytiques très simples (conditions de la forme de Lipschitz relatives à la distance d'une droite de la famille à un point fine, distance exprimée en fonction de l'angle avec une direction fine) pour qu'une famille de droites couvre simplement l'extérieur d'un cercle. Ils généralisent la notion d'enveloppe par l'introduction de deux sortes de points sur une droite; les points de contact et les points de giration (turning points); leur ensemble

forme un segment ou se réduit à un point. Ils étudient les ensembles de points du plan par où passent plus d'une droite de la famille (ceux par où il en passe un nombre pair forment un ensemble de mesure nulle). *J. Favard.*

***Vincensini, Paul. Les ensembles d'arcs d'un même cercle dans leurs relations avec les ensembles de corps connexes du plan euclidien.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 456–464. Casa Editrice Perrella, Roma, 1953.

This is an elaboration of an earlier note by the author [C. R. Acad. Sci. Paris **232**, 2075–2076 (1951); these Rev. **12**, 850], and contains several results on intersection properties of arcs on a circle, with applications to plane convex sets and to convex sets of directions in the plane. His principal result is the following: Suppose F is a finite family of plane connected sets, each four of which have a common secant. Suppose there is a line L none of whose parallels intersects more than one member of F. Then there is a secant common to all members of F. Related results are given by Santaló [Publ. Inst. Mat. Univ. Nac. Litoral **2**, 49–60 (1940); **3**, 202–210 (1942); these Rev. **2**, 261; **4**, 112] and Rademacher and Schoenberg [Canadian J. Math. **2**, 245–256 (1950); these Rev. **11**, 681]. Though differently phrased, the author's proof is essentially as follows: Suppose L is the y-axis and $F=\{C_1, \cdots, C_k\}$. For each i and j let e_{ij} be the set of all slopes of line segments joining a point of C_i to a point of C_j. Then the e_{ij}'s are connected sets of real numbers, each two of which have a common point, whence there is a point common to all of them. Thus there is a line M such that each two sets of F have a common secant parallel to M. The desired conclusion follows by considering the projections of the sets of F onto a line perpendicular to M, for each two of these have a common point. [Reviewer's notes: (a) By a more elaborate argument it is possible to prove the above theorem with "four" replaced by "three" in the hypothesis. (b) As Santaló has observed (in the first of his papers listed above), for each $n \geqq 2$ there are $n+1$ line segments in the plane which have no common secant even though each n of them have a common secant. From this it follows that the question near the end of §1 of Vincensini's paper admits an affirmative answer only when $h=0$.] *V. L. Klee* (Seattle, Wash.).

Bordoni, Piero Giorgio. Su certe proprietà generali dei raggi di girazione delle figure piane convesse. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 238–242 (1953).

Verfasser betrachtet einen homogenen ebenen convexen Körper mit Schwerpunkt G. Ist τ eine variable Achse durch G, ρ der bezügliche Gyrationsradius, d der Abstand der zwei mit τ paralleln Tangenten, dann werden die Grenzen von ρ/d betrachtet. Verwendet wird die generalisierte Translation welche bei dem Steinerschen Symmetrisierungsverfahren auftritt. *O. Bottema* (Delft).

Bordoni, Piero Giorgio. Limitazioni per il raggio di girazione baricentrale delle figure piane a contorno convesso. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 413–418 (1953).

Forsetzung von der oben referierten Arbeit. Verfasser zeigt dass die betrachtete Funktion ρ/d zwischen $(1/24)^{\frac{1}{2}}$ und $(1/12)^{\frac{1}{2}}$ liegt und gibt auch eine Verschärfung dieser Ungleichung an. *O. Bottema* (Delft).

Matsumura, Soji. Über Eilinien und Eiflächen. J. Osaka Inst. Sci. Tech. Part I. **3**, 35–38 (1951).

This paper is a series of observations concerning ovals in the plane and ovaloids in three and higher dimensions. The remarks on ovals are concerned with the angle θ which the affine normal makes with the ordinary normal. It is shown, for example, that if the angle θ is always the same for two points whose distance on the oval is a semicircumference, then the curvature is a periodic function with the semicircumference for a period. If two ovaloids in Euclidean 3-space are mapped on each other by making correspond points with the same directed normals, and if, under this mapping, the first two fundamental forms have a constant ratio, the surfaces are shown to be homothetic to each other.

The conclusion of section 6 appears to this reviewer to be false. It does not seem to follow from the work above in section 5, and its truth, together with the equations of section 5, yield the result that the curve is a circle, which certainly should have been stated if true. The paper is marred by a large number of typographical errors.

S. B. Jackson (College Park, Md.).

Matsumura, Soji. Remark to Misra's paper. J. Osaka Inst. Sci. Tech. Part I. **2**, 79–80 (1950).

The author proves a theorem concerning bounded plane regions such that the distance of any two exterior points is $\geqq 2r$ where $r>0$. Unfortunately such regions are nonexistent. *L. C. Young* (Madison, Wis.).

Hartman, Philip, and Wintner, Aurel. On pieces of convex surfaces. Amer. J. Math. **75**, 477–487 (1953).

Let $z(x, y)$ be a strictly convex function of class C^1 on a convex domain D. Then $p=z_x(x, y)$, $q=z_y(x, y)$ is a continuous 1-1 mapping of D on a (p, q)-domain E, and $h=xp+yq-z$ is of class C^1 and strictly convex on every convex subdomain of E; moreover, $x=h_p(p, q)$, $y=h_q(p, q)$. Even when $z(x, y)$ is analytic, it need not be of class C^2. A similar statement is proved for parallel surfaces to $z=z(x, y)$ and an example is given where a parallel surface is not of class C^2 and $z(x, y)$ is a polynomial.

H. Busemann (Los Angeles, Calif.).

Rinow, W. Bericht über die innere Flächentheorie A. D. Alexandrovs. Jber. Deutsch. Math. Verein. **56**, 77–87 (1953).

A brief report on A. D. Alexandrov's results on the intrinsic geometry of convex surfaces, dwelling mainly on the principal existence theorem. *H. Busemann.*

Algebraic Geometry

***Lefschetz, Solomon. Algebraic geometry.** Princeton University Press, Princeton, N. J., 1953. ix+233 pp. $5.00.

Da circa un secolo a questa parte, la geometria algebrica si è venuta consolidando ed arricchendo in modo prodigioso, sviluppandosi secondo esigenze e direttive molteplici—talora contrastanti—pertinenti all'algebra, all'analisi ed alla topologia: tanto che parrebbe utopistico il voler pervenire a dare nel breve volgere di poco più che 200 pagine un'esposizione introduttoria aggiornata e sufficientemente completa dell'argomento. Ora invece l'eminente scienziato di Princeton è brillantemente riuscito allo scopo con questo

magistrale volume, nel quale non mancano neppure sostanziosi contributi originali. In esso viene sopratutto approfondito il lato algebrico delle questioni, e l'uso che ivi vien fatto dell'algebra moderna vale a conferire alla trattazione grande generalità e nitidezza; ma anche i legami con l'analisi ne risultano chiariti, con l'uso metodico delle serie di potenze formali (considerate cioè a prescindere dalla loro convergenza), mentre il substrato geometrico e topologico trovasi sovente accennato con rapidi tocchi.

Il primo dei nove capitoli in cui l'opera è divisa riassume le nozioni ed i risultati algebrici preliminari occorrenti nel seguito, e concernenti gli anelli, i campi, gli ideali, i polinomi (con relativa decomposizione in fattori e teoria del risultante), 1 differenziali (sopra un campo di caratteristica zero). Nel cap. II, premessi alcuni richiami sugli spazi affini e proiettivi, vengono introdotte le varietà algebriche (effettive) sopra un corpo K commutativo ed i loro ideali, con le nozioni di punto generale, dimensione, corpo di funzioni, studiandone le proiezioni e dando qualche proprietà degli spazi tangenti e delle intersezioni nei casi più semplici (teorema di Bézout, ecc.).

Il cap. III passa ai problemi concernenti le trasformazioni algebriche fra varietà algebriche, e specialmente le trasformazioni birazionali e razionali, con l'uso delle varietà-prodotto introdotte in modo astratto. Si ottengono così varie rappresentazioni di tipo monoidale per le varietà algebriche e per le corrispondenze fra esse; vengono inoltre approfondite proprietà relative all'irriducibilità delle sezioni iperpiane d'una varietà o concernenti varietà definite sopra un campo di caratteristica non nulla.

Nel cap. IV viene schizzata una teoria generale delle serie di potenze formali e delle varietà algebroidi, di cui vengono quindi fatte applicazioni allo studio delle proprietà locali delle varietà algebriche, con particolare riguardo alle proprietà topologiche nel caso del corpo complesso. Mentre in ciò che precede viene generalmente ammesso che il corpo K di base possa essere qualsiasi, nella parte restante del volume tale corpo è sempre supposto algebricamente chiuso e di caratteristica zero. In una breve appendice viene poi data una suggestiva spiegazione del perchè tali restrizioni bastino ad assicurare la validità di tutte le proprietà algebriche inerenti al caso in cui K sia il corpo complesso.

Nel cap. V, lo studio delle curve algebriche sopra K viene proseguito con l'opportuno intervento degli anelli $K[[x^{1/n}]]$ $(n=1, 2, \cdots)$ delle serie di potenze formali in $x^{1/n}$ a coefficienti in K e dei relativi corpi quozienti $K((x^{1/n}))$, nonchè del corpo $K\{x\}$ unione di questi ultimi. Così, ad esempio, il teorema di Puiseux rientra nella proprietà di chiusura algebrica di $K\{x\}$, e le nozioni di ramo, di posto e di valutazione, di punto multiplo su di una curva piana vengono del pari introdotte ed approfondite con l'uso degli sviluppi in serie. Viene quindi studiato il comportamento dei posti di fronte alle trasformazioni razionali e birazionali, ed i precedenti risultati vengono poi estesi alle curve immerse in uno spazio proiettivo di dimensione qualsiasi, studiando anche le singolarità delle proiezioni piane generiche di queste ultime, e pervenendo infine—con procedimento puramente algebrico—a due teoremi classici sulla riduzione delle singolarità di una curva.

Il successivo cap. VI tratta delle prime questioni concernenti le serie lineari g_n^r (serie complete, operazioni di somma e differenza, modelli proiettivi, ecc.), con applicazioni alla nozione di classe di una curva piana e ad un primo studio delle curve razionali, ellittiche ed iperellittiche. Esso culmina col teorema di Riemann-Roch, ottenuto attraverso la costruzione delle serie lineari complete mediante le aggiunte ed il teorema di riduzione di Noether, e col complemento fornito dalla legge di reciprocità di Brill-Noether.

Segue (nel cap. VII) lo studio dei differenziali abeliani, fatto senza mai introdurre i relativi integrali, la considerazione dei quali dev'essere qui evitata in quanto non si suppone la continuità di K. Stabilito il legame fra differenziali di 1ª specie e serie canonica (di cui viene fatta applicazione alla teoria delle curve razionali, ellittiche ed iperellittiche, ed allo studio dei modelli canonici), viene analizzato il comportamento dei differenziali delle varie specie e dei loro residui di fronte alle trasformazioni birazionali e razionali. Si determinano quindi certi differenziali elementari di 2ª e 3ª specie, traendone varie consequenze, e si ottiene il legame fra serie jacobiane e serie canonica, pervenendo anche alla formula di Zeuthen ed al suo significato funzionale, nel caso di una trasformazione razionale fra curve.

Il cap. VIII stabilisce il teorema di Abel ed il suo inverso, ed approfondisce lo studio delle corrispondenze algebriche fra curve. Si giunge così alla formula di Zeuthen nel caso generale, ed alle prime proprietà delle corrispondenze a valenza (quelle a valenza nulla sono chiamate brevemente corrispondenze lineari); inoltre, nel caso di due curve sovrapposte, si stabilisce il classico principio di corrispondenza di Cayley-Brill-Hurwitz (il nome del primo di questi tre autori è qui sempre scritto con un'*i* in luogo della prima *y*, e inoltre vien taciuto quello del terzo).

L'ultimo capitolo fornisce la prima [o la seconda] dimostrazione puramente algebrica del teorema di Severi affermante la finitezza della base per la totalità delle curve tracciate su di una superficie algebrica, di fronte all'equivalenza algebrica [una dimostrazione algebrica di altro tipo è state data quasi simultaneamente da A. Néron nel lavoro recensito sotto]. All'uopo vengono ammessi senza dimostrazione vari risultati già ottenuti per via algebrica da altri autori, come il teorema sullo scioglimento delle singolarità, quello sull'algebricità del sistema formato dalle curve di dato ordine giacenti su di una superficie assegnata, ed il criterio aritmetico di Severi perchè una curva virtuale su di una superficie risulti effettiva; si può così seguire passo passo la via originalmente percorsa da Severi, poggiante però sul teorema—ottenuto con metodo non algebrico da Picard—che afferma la finitezza del numero ρ_3 dei differenziali doppi indipendenti di 2ª specie su di una superficie, modulo le forme differenziali chiuse. L'argomentazione di Picard è qui sostituita con altra di carattere algebrico, che porta fra l'altro a riottenere un lemma di Castelnuovo con un nuovo procedimento che evita ogni considerazione di continuità. Sono poi rapidamente indicati i legami del suddetto teorema di Severi con la topologia, ed alcune fra le applicazioni che da esso discendono alla teoria delle corrispondenze fra curve algebriche.

Il volume, anche se talvolta un po' stringato, è estremamente ben scritto. Esso si chiude con una breve bibliografia, ed un particolareggiato indice alfabetico. *B. Segre.*

Néron, André. Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps. Bull. Soc. Math. France 80, 101–166 (1952).

If U is an algebraic irreducible variety, we denote by $G(U)$ the additive Abelian group of the U-divisors and by $G_1(U)$, $G_a(U)$ the groups of the U-divisors which, on U, are linearly or algebraically equivalent to zero respectively.

If K is a field of definition of U, an element of $G(U)$ is said to be rational over K if it is invariant with respect to every automorphism of the algebraic closure $\bar{K}$ of K which preserves the elements of K. The totality of the rational divisors of U is a subgroup $G(U, K)$ of $G(U)$; and we shall also consider the groups $G_1(U, K) = G_1(U) \cap G(U, K)$, $G_0(U, K) = G_0(U) \cap G(U, K)$, $\gamma(U, K) = G_0(U, K)/G_1(U, K)$.

When U is a non-singular curve C of genus g, containing a rational divisor of degree g, the minimum number r of generators of $\gamma(C, K)$ has been called by Poincaré the rank of C in K; and a reduced rank r_0 can be defined also when C is singular. The finiteness of r was proved first by Mordell [Proc. Cambridge Philos. Soc. **21**, 179–192 (1922)] for $g=1$ and assuming K to be the field of rational numbers, and then by Weil [Acta Math. **52**, 281–315 (1929)] for arbitrary g and on supposing that K is any finite algebraic field. This remarkable result is now extended to the case when, g being arbitrary, K has any characteristic p and can be generated on its prime subfield by a finite number of elements. The extension is obtained here by associating to C a suitable algebraic variety U, by showing then that $\gamma(C, K)$ is isomorphic to a subgroup of the direct product of $G(U, K)/G_0(U, K)$ and of the group of the rational points of a certain Abelian variety (related to the Picard variety of U), and finally by proving the existence of finite bases for the last two groups.

The results are attained throughout by purely algebraic methods, with the intervention of the notions of complexity (Northcott) or of height (Weil) of a point over a field of characteristic zero, and by the use of an "infinite descent". Thus algebraic proofs and extensions of two fundamental theorems by Severi and by Weil are given; the existence of a finite basis for $G(U, K)/G_0(U, K)$, obtained here for arbitrary U and K, was proved first by Severi with transcendental methods [Math. Ann. **62**, 194–225 (1906); Ist. Veneto Sci. Lett. Arti. Parte II. **75**, 1121–1162 (1916)], in the case when K is the complex field. Around these important contributions there are many other interesting results, like those of a preliminary chapter concerning intersections (both in the regular and in the irregular case), correspondences, equivalence theory, Jacobi and Abelian varieties.

The last chapter deals in detail with the question of determining the rank of a curve variable in certain linear systems, and shows its connection with the problem of finding the base-number for the Severi group of a convenient algebraic variety. It is proved that, if C is a curve of genus $g \geqq 1$ over a field K and $(A_1, A_2, \cdots, A_m)$ denotes a set of m generic independent points of C, then the reduced rank r_0 of C in $K(A_1, \cdots, A_m)$ is $\geqq m$. Moreover, curves having the rank $r \leqq 4g+4$ or $r \geqq 3g+6$ in certain fields are constructed, and the conditions for the first ones to have $r = r_0 = 4g+4$ are given. Thus, e.g., if we fix eight points on a plane over any field K, such that neither three of them are collinear nor six of them lie on a conic, then through those points there is an infinity of plane cubics having the reduced rank $\geqq 9$ in K; moreover, through six points subject to the same conditions there is an infinity of cubics of reduced rank $\geqq 10$ in K. *B. Segre* (Rome).

Masotti Biggiogero, Giuseppina. Sopra il teorema di Reiss. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 187–222 (1951).

Le théorème de Reiss, relatif à une courbe algébrique orientée C et à une droite orientée D, exprime que la somme $\sum 1/(R \sin^3 \varphi)$ étendue à tous les points communs à C et D (R = rayon de courbure de C; φ = angle de C et D) est nulle. Ce théorème a été récemment généralisé par B. Segre [Ann. Mat. Pura Appl. (4) **26**, 1–26 (1947); ces Rev. **10**, 397].

L'auteur donne ici une nouvelle démonstration du théorème de Reiss basée sur la signification analytique de chaque terme de la somme et sur l'étude de la limite d'une somme de termes relatifs à des points d'une droite D mobile qui tendent vers un même point limite, de façon que le cas où D est tangente à C, ou passe par des points singuliers de C soit également envisagé. Par exemple: Lorsque D est une tangente en un point simple, le terme de Reiss relatif aux deux points confondus au contact est fini. Il est nul lorsque et seulement lorsque le point de contact est sextactique. On a aussi le théorème: Le terme de Reiss relatif à une branche superlinéaire Γ d'ordre ν et une droite D passant par son origine, est le produit par ν du terme de Reiss relatif à D et à la branche linéaire lieu des barycentres des sections de Γ par les parallèles à D. Ces propriétés sont établies en utilisant les développements de Puiseux. La démonstration comporte cependant une part transcendante: l'auteur fait varier C dans un faisceau $\lambda C + G = 0$, où G est une courbe du même ordre que C, décomposée en droites, et elle remarque que la somme de Reiss est une fonction rationnelle de λ dépourvue de pôles, donc constante, et nulle pour $\lambda = 0$.

Lorsqu'on transforme projectivement la figure, le rapport des termes de Reiss en des points homologues est un invariant de la transformation, l'invariant de Study-Bompiani [Bompiani, Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **22**, 483–491 (1935)]. Ceci permet à l'auteur de donner une condition nécessaire et suffisante pour qu'une courbe algébrique soit invariante dans une homologie harmonique. Elle en tire diverses propriétés géométriques des cubiques, des quartiques, et des courbes algébriques C^n ayant un point $(n-2)$uple.

Le mémoire se termine par la démonstration d'une relation analogue à celle de Reiss, concernant les éléments différentiels du 3ième ordre, obtenue à partir de celle de Reiss par une dérivation adroitement conduite. Il comporte également un historique bibliographique de la question.
L. Gauthier (Nancy).

Teixidor, J. Über die Umkehrung des Theorems von Reiss. Arch. Math. **4**, 225–229 (1953).

On a plane (x, y), consider n linear elements $x = x_i(y)$ $(i = 1, 2, \cdots, n)$, where the right hand sides are functions defined and differentiable at least $n+1$ times in a neighbourhood of $y=0$. A well known theorem, due to Reiss, asserts that, if those elements lie on an algebraic curve of order n, then $\sum_{i=1}^{n} x_i''(0) = 0$; and a similar relation holds if we substitute for the line $y=0$ any line of the plane. Conversely, if such a relation holds for all the lines $y = mx + h$ in the neighbourhood of $y=0$, then the n elements belong to an algebraic curve of order n.

The reciprocal result, due to S. Lie and F. Engel, is proved here in a very simple way, by showing that, as a consequence of Reiss' conditions, the functions x_i satisfy the identities

$$\sum_{i=1}^{n} \frac{d^{k+1}[x_i(y)]^k}{dy^{k+1}} = 0 \quad (k = 1, 2, \cdots, n).$$

Hence $\sum_{i=1}^{n} [x_i(y)]^k$ is a polynomial of degree $\leqq k$ in y, and so $x = x_i(y)$ are the solutions of an equation of the form $x^n + P_1(y)x^{n-1} + \cdots + P_n(y) = 0$, where P_k is a polynomial of degree $\leqq k$ in y. *B. Segre* (Rome).

Manara, Carlo Felice. Le condizioni perche due curve gobbe siano omologiche rispetto ad un centro assegnato. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 15–22 (1951).

The paper is devoted to a proof, under restricted conditions, of the theorem: The necessary and sufficient condition that two curves γ and γ', lying on the same cone of vertex 0 and such that any branch of γ through 0 is tangent to a branch of γ' through 0, shall be images under a homology with center at 0 is that the differential invariants J (of Memke-Segre) defined by the pairs of branches of γ and γ' through 0 and certain projective invariants defined by the intersections of γ and γ' with their common chords through 0 shall all have the same value. *G. B. Huff.*

Tibiletti, Cesarina. Gruppo concentrato di intersezioni di due curve algebriche. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 29–47 (1951).

Tibiletti, Cesarina. Il teorema generale dell' $A\phi+B\psi$ dedotto da un computo di costanti. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 300–304 (1951).

If $\phi=0$ and $\psi=0$ are curves in the complex projective plane with intersection multiplicity i at a point P, the author describes the concept of a curve, $f=0$, "passing through the group of intersections of $\phi=0$ and $\psi=0$ which cluster at P" by a system S_P^i of i linearly independent linear conditions on the coefficients of f. The curves $\phi=0$ and $\psi=0$ are said to have in common a "set G of i points concentrated at P" and a curve whose coefficients satisfy the conditions S_P^i is said to "pass through" the set G. In the first paper the properties of these concepts are studied by the classical methods of algebraic geometry. The second paper gives a statement of the general $A\phi+B\psi$ theorem of Noether in the language of the first, and sketches a proof by an argument depending on counting constants.

G. B. Huff (Athens, Ga.).

Godeaux, Lucien. Construction de surfaces canoniques de diviseur deux. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 653–665 (1953).

Construction de surfaces dont les sections hyperplanes constituent le système canonique, images d'involutions du second ordre privées de points unis et par conséquent de diviseur deux. *Résumé de l'auteur.*

***Gauthier, Luc. Sulle congruenze d'ordine uno di rette di S_4 la cui focale propria è irriducibile.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 329–337. Casa Editrice Perrella, Roma, 1953.

Dans un espace projectif à quatre dimensions on considère un système algébrique ∞^3 de droites qui remplit tout l'espace, c'est-à-dire une congruence de droites; on sait alors que sur chaque droite de la congruence il y a trois foyers, dont le lieu est en général une surface algébrique F. L'auteur s'occupe ici des congruences de l'ordre 1, dans l'hypothèse que sur toute droite générale de la congruence les trois foyers soient distincts et que la surface F lieu des foyers soit irréductible. La recherche des types différents de ces congruences revient à la détermination de toutes les surfaces algébriques irréductibles de l'espace S_4 qui possèdent un seul point triple apparent. Cette détermination a été faite par F. Severi en 1901, en supposant encore que F possède seulement des singularités ordinaires. Comme l'auteur se place ici dans le cas le plus général, après avoir retrouvé les solutions déjà données par F. Severi, il trouve encore une autre solution, c'est-à-dire une surface F^7 de S_4 que l'on peut obtenir de la manière suivante. On considère dans un espace S_5 la surface rationnelle F^7 representée sur le plan par le système linéaire de toutes les courbes du 4e ordre passant par neuf points O_i; la cubique définie par les points O_i représente une cubique plane γ de la surface F^7; en projetant F^7 sur un S_4 d'un point du plan de γ on obtient dans S_4 une surface F^7 possédant une droite triple Δ et un système ∞^2 de courbes du 4e ordre et de la 2e espèce dans les hyperplans passant par Δ, qui est une trisécante commune de toutes ces courbes; les autres trisécantes de ces mêmes courbes forment une congruence de droites de l'ordre 1. Dans sa recherche l'auteur tient compte soit de la surface focale singulière, qui est formée par des plans coupant F selon des courbes d'ordre ≥ 4, soit de la surface focale spéciale, qui est formée par des plans coupant F selon des cubiques.

E. G. Togliatti (Gênes).

Guazzone, Stefano. Sulle ipersuperficie di S_k e di ordine s che appartengono alla ipersuperficie generale di ordine n di S_r $(r>k)$. Rend. Sem. Mat. Univ. Padova **21**, 243–251 (1952).

La détermination de conditions nécessaires et suffisantes pour que l'hypersurface F^n_{r-1} de S_r contienne des S_k en nombre fini ou infini a fait l'objet d'une publication récente de B. Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 261–265, 341–346 (1948); ces Rev. **10**, 207, 566]. L'auteur examine ici l'appartenance à F^n_{r-1} de S_r d'hypersurfaces F^s_{k-1} de S_k $(k<r;\ 2\leqq s\leqq n-2)$. En posant:

$$\delta=(r-k)(k+1)-\binom{n+k}{k}+\binom{n+k-s}{k}+\binom{k+s}{k}-1$$

la condition nécessaire et suffisante pour que la forme F^n_{r-1} générique de S_r contienne des F^s_{k-1} est que $\delta\geqq 0$. Si $\delta=0$, F^n_{r-1} contient un nombre fini de F^s_{k-1}. Si $\delta>0$, les F^s_{k-1} situées sur F^n_{r-1} forment un système algébrique de dimension δ, pur, irréductible. A toute F^s_{k-1} correspond une F^{n-1}_{k-1} résiduelle de la section de F^n_{r-1} par S_k. L'inégalité obtenue ne se réduit pas à celle de B. Segre pour $s=1$: cela tient à ce que par un S_{k-1} passent tous les S_k d'une étoile donnant lieu à un système linéaire de F^{n-1}_{k-1} résiduelles, et non à une correspondance biunivoque $(s\leftrightarrow n-s)$ comme pour $s\neq 1$. *L. Gauthier* (Nancy).

Burniat, Pol. Surfaces algébriques à système canonique pur dégénéré. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 1030–1043 (1952).

Il existe des surfaces sur lesquelles le système canonique pur présente les divers types de dégénérescence des systèmes linéaires. Voir la bibliographie de cette question dans mon compte-rendu [ces Rev. **10**, 323] du mémoire de G. Pompilj sur le même sujet [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 539–544 (1948)].

Dans le présent mémoire, l'auteur généralise le résultat de Pompilj par une étude systématique des plans doubles définis par un faisceau de courbes de genre 2. Lorsque l'involution I_2 qui définit le plan double est engendrée par les g_2^1 des courbes du faisceau, on obtient six classes de plans doubles, réguliers, dont la première est identique à celle de Pompilj. Les genres sont:

$$p_a=p_g,\quad p^{(1)}=4p_g-5+i,\quad P_2=5p_g-5+i,$$

où $i=0$ dans les classes I, II, III; $i=1$ dans les classes IV et V; $i=2$ dans la classe VI. p_g est de même parité que la

classe. Le système canonique est composé d'une courbe fixe de genre p_g-2+i, et de (p_g-1) courbes de genre 2 variables dans un faisceau. Le système bicanonique est composé au moyen de I_2.

Lorsque l'involution I_2 qui définit le plan double est engendrée par des γ_2^1 elliptiques appartenant aux courbes du faisceau, on obtient deux classes de plans doubles, réguliers, dont les genres sont donnés par les mêmes expressions, avec $i=0$ dans la première classe, et $i=1$ dans la seconde classe. p_g est un entier arbitraire. Les courbes de branchement, d'ordre $6p_g$, présentent $(8-i)$ points multiples d'ordre $2p_g$ et, dans la première classe un point d'ordre $2p_g-2$, dans la seconde classe un doublet d'ordre $2p_g-1$. Dans ces deux classes le système bicanonique n'est pas composé avec I_2. *L. Gauthier* (Nancy).

Muracchini, Luigi. Sulle varietà V_3 analitiche pluririgate. Boll. Un. Mat. Ital. (3) **8**, 138–144 (1953).

Les seules variétés analytiques à trois dimensions, contenant un système ∞^2 de droites, tel que par un point passent m droites dont trois ne sont jamais copolanaires, sont: les quadriques de S^4 $(m=\infty)$ les cubiques générales de S^4 $(m=6)$, les variétés quartiques de S^5, bases d'un faisceau de quadriques $(m=4)$, les V_6 de S^7 définies par C. Segre comme images des triples de points de trois droites $(m=3)$. Si $m=2$ pour tout S^n où $n>10$ les V^3 sont lieux de ∞^1 quadriques. Ce résultat s'obtient de l'étude des formes de Pfaff qui apparaissent dans les équations définissant le déplacement du repère. *B. d'Orgeval* (Alger).

***Chisini, Oscar. Piani multipli e questioni topologiche connesse.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 78–87. Casa Editrice Perrella, Roma, 1953.

Exposé de l'état actuel de la théorie des plans multiples algébriques et de leurs courbes de diramation, avec particulière attention à la théorie des tresses due à l'A. et développée par ses disciples. *B. d'Orgeval* (Alger).

Spampinato, Nicolò. Sull'estensione del teorema di Lüroth dall'S_1 complesso ad un S_1 ipercomplesso. Atti Accad. Ligure **9** (1952), 36–39 (1953).

Dans toute algèbre A d'ordre m dotée de module et commutative, une série J_n^I de la droite projective liée à A est une involution. La démonstration se fait simplement dans le cas où A est irréductible en remarquant que la matrice composée à partir de celles des coefficients de l'équation qui détermine le groupe de J contenant un point donné doit être de caractéristique maxima. *B. d'Orgeval* (Alger).

***Martinelli, Enzo. Alla ricerca di nuovi integrali invarianti sulle varietà algebriche.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 398–406. Casa Editrice Perrella, Roma, 1953.

In order to study harmonic integrals on a complex manifold, one has to introduce a metric which is not, in general, associated with the manifold in any relevant way. This paper discusses other methods of defining invariant integrals which do not depend on the arbitrary selection of a metric. The author considers an invariant notion of conjugacy between pairs of p-forms (instead of between a p-form and a $(2m-p)$-form, as in the case of harmonic forms), and the invariant forms considered are those closed forms whose conjugates are also closed. The invariant forms are shown to be just the forms which can be written as sums $\sum_i \phi^{(r-s,s)}$, where $\phi^{(r,s)}$ is a closed form of type (r, s). [Properties of forms of this type have been considered, by means of the theory of faisceaux, by P. Dolbeault [C. R. Acad. Sci. Paris **236**, 175–177 (1953); these Rev. **14**, 673] and in various mimeographed notes of lectures given at Princeton University by K. Kodaira and D. C. Spencer.]

W. V. D. Hodge (Cambridge, England).

Severi, Francesco. Un'osservazione sul limite d'applicabilità della formula di postulazione per una varietà algebrica. Atti Accad. Ligure **9** (1952), 23–28 (1953).

The postulation, for primals of order l, of an irreducible variety V_d in S_r, or the Hilbert characteristic function of its associated prime ideal, is, for $l \geqq l_0$, a polynomial in l of order d. In this paper the author is concerned with cases in which it is possible to assign an upper bound for l_0. In particular, if V_d and all its generic sections by linear spaces of various dimensions are arithmetically normal in Zariski's sense, $l_0 \leqq m-d-1$. The same result holds for arithmetically normal surfaces, provided only that they are non-singular.

D. B. Scott (London).

Chern, Shiing-shen. On the characteristic classes of complex sphere bundles and algebraic varieties. Amer. J. Math. **75**, 565–597 (1953).

The reviewer [Proc. London Math. Soc. (3) **1**, 138–151 (1951); these Rev. **13**, 380] has proved that the characteristic cohomology classes of a non-singular algebraic variety V_m are the duals of the canonical classes of Todd and Eger, in case V_m is a complete intersection, but he has only indicated a possible method of proving the result for general varieties. The method is an indirect one, starting from a knowledge of the characteristic classes of a projective space and using an adjunction formula. The present paper considers a direct approach to the problem, and has as its main object to show that dual characteristic homology classes of V_m contain algebraic cycles.

The r^d characteristic cohomology class C_r of V_m is the primary obstruction cohomology class of the bundle $B^*_{m,r}$ over V_m where fibres are formed by the ordered sets of r independent tangent vectors to V_m at a point. The author introduces two new bundles over V_m: the bundle $\bar{B}_{m,r}$ where fibres consist of the sequences of a vector e_1, a bivector $e_1 \wedge e_2$, $\cdots$, a multi-vector $e_1 \wedge e_2 \wedge \cdots \wedge e_r$ at a point of V_m, and the bundle $B_{m,r}$ where fibres consist of a tangent direction L_0, a tangent plane L_2, etc., where $L_0 \subset L_1 \subset \cdots \subset L_r$. Natural projections

$$B^*_{m,r} \xrightarrow{m} \bar{B}_{m,r} \xrightarrow{p} B_{m,r} \xrightarrow{q} V_m$$

are easily defined and each adjacent pair is a fibre bundle and base space with the associated projection as bundle projection, and with the general complex linear group G_m as group. The fibre of the bundle p: $\bar{B}_{m,r} \to B_{m,r}$ is the direct product of r groups Γ isomorphic to the multiplicative group of non-zero complex numbers. The characteristic class of this bundle can be computed in terms of the characteristic class of a complex linear bundle over $B_{m,r}$, and since $B_{m,r}$ is algebraic these are duals of divisors on $B_{m,r}$ [Weil, Conference on algebraic geometry, Chicago, 1949]. It is then possible to establish a relation between the characteristic class of the bundle p: $\bar{B}_{m,r} \to B_{m,r}$ and the images $q^*(C_i)$ of the classes C_i induced by the projection q. The main theorem is then obtained by applying the process of integrating over the fibre of q: $B_{m,r} \to V$, a process which, in the present instance, is equivalent to replacing a cocycle of $B_{m,r}$ by a

dual cycle, projecting on V_m, and then taking the dual cocycle on V_m. *W. V. D. Hodge* (Cambridge, England).

Differential Geometry

***Thomas, Paul D. Conformal projections in geodesy and cartography.** U. S. Department of Commerce, Coast and Geodetic Survey, Special Publication no. 251. United States Government Printing Office, Washington, D. C., 1952. ix+142 pp. (1 plate). 60 cents.

This monograph presents a résumé of established characteristics of conformal projections of a spheroid on a plane. The material is developed and coordinated using the conformal behavior of analytic functions of a complex variable. The method is identical in principle to that presented recently by R. König and K. H. Weise [Mathematische Grundlagen der höheren Geodäsie und Kartographie, Bd. 1, Springer, Berlin, 1951; these Rev. **13**, 157]. This previous work and the essential duplication in content is not acknowledged or mentioned by the author. Following a brief description of the primary conformal projections, an elementary discussion of the theory of analytic functions of a complex variable is given. This is applied in a succeeding general development of the differential geometry of surfaces and curves of concern in map projections. These general principles are applied in the final sections to develop systematically the several projections and their differential characteristics. Not only is a coordinated theory presented, but the correlation to practice in map projections is well documented. *N. A. Hall* (Minneapolis, Minn.).

Gougenheim, A. Emploi des projections conformes en cartographie. Bull. Géodésique **1953**, 7–37 (1953). (German, English, Spanish and Italian summaries)

Lecture given in June 1952.

Gallarati, Dionisio. Alcune osservazioni sulle curve sferiche ed una nuova caratterizzazione della sfera. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 238–241 (1952).

Let F be a [three times continuously differentiable] surface in real euclidean 3-space. Let L be a curve on F which is three times [continuously] differentiable and whose curvature never vanishes [additions by the reviewer]. Let $\bar{L}$ denote the binormal spherical image of L and let L^* be the locus of the centers of curvature of L. Then F is a sphere of radius R if and only if the arc length of L^* is R-times that of $\bar{L}$ for every L. If F is a sphere about the origin, then $\bar{L}$ is identical with the projection of L^* from that point on the unit sphere. *P. Scherk* (Saskatoon, Sask.).

***Müller, Hans Robert. Sulla proiezione cinematica nello spazio euclideo.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 407–411. Casa Editrice Perrella, Roma, 1953.

Verfasser gibt eine kurze Übersicht von den von Study, Blaschke und Grünwald herrührenden Betrachtungen über kinematische Abbildungen und fügt einige Ergänzungen zu. *O. Bottema* (Delft).

Müller, Hans Robert. Flächenläufige Bewegungsvorgänge im elliptischen Raum. II. Monatsh. Math. **57**, 129–133 (1953).

[Fortsetzung von Monatsh. Math. **57**, 29–43 (1953); diese Rev. **14**, 1014.] Bei einem geschlossenen zweigliedrigen Bewegungsvorgang beschreibt ein Punkt P des Gangraums eine geschlossene Bahnfläche die ein Raumstück begrenzt. Verfasser berechnet den Inhalt dieses Raumstücks unter Zugrundelegung einer elliptischen Massbestimmung und erhält so eine Erweiterung der von Blaschke für den euklidischen Raum aufgestellte Formel. Der Inhalt ist eine quadratische Funktion der Koordinaten von P. *O. Bottema* (Delft).

Lauffer, Rudolf. Analytische Kurven auf einer Fläche 2. Ordnung. Math. Nachr. **9**, 301–306 (1953).

Let A be a 4×4 matrix with complex coefficients; $A=A'$; $|A|=1$. Let $X, Y, \cdots$ denote column vectors with four elements. Thus $(X|Y)=X'AY$ is a symmetric bilinear form and $(X|X)=0$ is the equation of a non-degenerate quadric Φ in complex projective 3-space. Given an analytic curve C: $U=U(\tau)$ on Φ. Suppose $(dU/d\tau|dU/d\tau)$ and the determinant $(U, dU/d\tau, d^2U/d\tau^2, d^3U/d\tau^3)$ do not vanish in some region of the complex τ-plane. Then a weight function $\lambda(\tau)$ and a parameter $t(\tau)$ exist such that $V=\lambda U$ satisfies the conditions $(\dot{V}|\dot{V})=(V, \dot{V}, \ddot{V}, \dddot{V})=1$. Here dots indicate differentiation with respect to t. Put $J=(\ddot{V}|\ddot{V})$ and let W denote the pole of the osculating plane of C; $(W|W)=-1$. Then $\dddot{V}=-\frac{1}{4}JV-J\dot{V}+W$ and $\dot{W}=-V$. Thus J is the only independent differential invariant of C. A projectivity which maps Φ onto itself transforms J into $\pm J$. By means of V, $\dot{V}$, $\ddot{V}$, W an invariant moving polar simplex of C is readily constructed. Formulas for J and $\dot{V}$ are computed if C is given in the form $U=U(\tau)$ and if Φ is provided with a special parametrization. The skew W-curves on Φ are determined. *P. Scherk* (Saskatoon, Sask.).

Şemin, Ferruh. Sur une propriété caractéristique des cyclides de Dupin et des surfaces de révolution. Rev. Fac. Sci. Univ. Istanbul (A) **18**, 288–297 (1953). (Turkish summary)

A Darboux curve on a surface S is a curve whose osculating spheres are always tangent to S. Consider the D-curves passing through a regular, non-umbilic point P on S, and let C be the locus of their centers of curvature at P. Blaschke [Math. Z. **27**, 150–153 (1927)] and the present author [Rev. Fac. Sci. Univ. Istanbul (A) **17**, 351–383 (1952); these Rev. **14**, 685], among others, have studied the surfaces for which C must be a plane curve whenever P is a regular, non-umbilic point. In this paper the author shows that if the curve C is to be a spherical curve whenever P is a regular, non-umbilic point of S, then S must be either (1) a cyclide of Dupin (including the torus), or (2) a surface of revolution other than a quadric surface of revolution. For the points of certain special curves on these surfaces, the curve C will be planar. Omitting these points, the author describes in case (1) the ellipse, and in case (2) the cubic plane curve from which the curve C can be obtained by inversion in a unit sphere. Further, if in case (1) the surface S has no double points and the curve C for the regular, non-umbilic point P is not planar, then the point P itself, the centers of the two lines of curvature through P (which are circles), and the centers of the two circles of Villarceau through P all lie on a sphere. *A. Schwartz* (New York, N. Y.).

Grove, V. G. On generalized curvatures. Univ. Nac. Tucumán. Revista A. **9**, 47–55 (1952). (Spanish)

Partant de la considération de la projection oblique Γ, suivant une direction λ, d'une courbe C d'une surface sur le plan tangent en l'un de ses points, et de la notion qui en

résulte de courbure relative de C, Springer et Dekker ont été conduits à envisager une courbure dite axiale [Springer, Bull. Amer. Math. Soc. **51**, 686–691 (1945); ces Rev. **7**, 172], et une courbure hypergéodésique ou supergéodésique [Dekker, ibid. **55**, 1151–1168 (1949); Pacific J. Math. **1**, 53–57 (1951); ces Rev. **11**, 394; **13**, 383]. Dans l'article actuel l'auteur introduit ces notions par une nouvelle méthode, n'exigeant pas l'emploi des courbes axiales, hypergéodésiques ou supergéodésiques utilisées par Springer et Dekker. Après avoir étudié tensoriellement la courbure relative d'une courbe d'une surface, il définit, sur la surface, une famille de courbes satisfaisant à une équation différentielle de forme déterminée qu'il appelle courbes coniques. Cette famille de courbes renferme comme cas particuliers les courbes envisagées par Springer et Dekker, et sa considération permet de retrouver les courbures axiales, hypergéodésiques et supergéodésiques, en même temps qu'elle livre une nouvelle espèce de courbure, plus générale que les courbures indiquées. *P. Vincensini* (Marseille).

Gorowara, K. K. On spherical representation of generators of a ruled surface. Ganita **4**, 85–91 (1953).

L'auteur applique le calcul vectoriel dualistique à la recherche de propriétés de la représentation sphérique des génératrices d'une surface réglée. A coté de résultats obtenus par Mishra et Chariar relatifs à la notion de "skewness" de la distribution des génératrices d'une surface réglée, il fait connaître au sujet de cette notion un certain nombre de résultats nouveaux. Ainsi, la torsion de la représentation sphérique des génératrices de la surface (A_1) est égale à la "skewness" de distribution des génératrices de (A_2) multipliée par la courbure de la représentation sphérique des génératrices de (A_1). Ce résultat admet des cas particuliers intéressants. Par exemple, si θ désigne l'angle que fait la normale principale de la représentation sphérique des génératrices de (A_1) avec la normale à la sphère, et si $\theta=\pi/4$, la représentation sphérique de (A_2) est une géodésique; en outre, dans ce même cas particulier $(\theta=\pi/4)$, les paramètres de distribution λ_1, λ_2, λ_3 pour trois rayons correspondants de (A_1), (A_2), (A_3) sont en progression arithmétique. De même, si la ligne de striction de (A_1) est une asymptotique, la "skewness" relative à une génératrice de (A_1) est égale à la cotangente de l'angle que fait cette génératrice avec la ligne de striction. Ou encore, si la ligne de striction est de courbure, la "skewness" est l'opposée de la tangente de l'angle formé par la génératrice et la ligne de striction. Et aussi: si la ligne de striction est à la fois ligne de courbure et ligne géodésique, la représentation sphérique des génératrices de (A_2) est une géodésique.

P. Vincensini (Marseille).

Bukovics, E. Eine topologische Invariante von sechs Linienelementen zweiter Ordnung eines Punktes der Ebene. Monatsh. Math. **57**, 117–128 (1953).

The ∞^2 differential elements of the second order, E_2, through a point of a plane considered with respect to point transformations of class C^2 can be mapped as points of a rational cubic cone in a projective space of four dimensions S_4; the point-transformations (up to and including the neighborhood of the second order) are mapped on the collineations of S_4 leaving the cone invariant (its vertex, corresponding to the point, is fixed). Leaving aside invariants depending only on elements of the first order E_1, the minimum number of E_2 having an invariant is six, and there is just one essential invariant of the second order of six E_2 with same center. Analytic expressions of the invariants are given with an accurate bibliography. *E. Bompiani.*

Segre, Beniamino. L'élément linéaire projectif d'une congruence quadratique de droites. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 481–489 (1953).

Soit une congruence quadratique de droites K, c'est-à-dire une congruence dont tous les rayons appartiennent à une hyperquadrique Ω dans l'espace S_n, soient P et Q les surfaces focales; l'auteur définit en tout point de P et Q trois tangentes principales, d'où il déduit les courbes principales sur ces surfaces et la distribution de tous les rayons de K en trois familles ∞^1 de surfaces, dites réglées principales de K. Il peut alors caractériser les congruences K dont une ou les deux transformées de Laplace appartiennent à l'hyperquadrique Ω qui contient K. Pour une congruence quadratique dans le cas général, l'auteur définit un élément linéaire projectif, analogue aux éléments linéaires projectifs que Fubini a attachés dans S_3 à une surface F non réglée ou à une congruence de droites.

Si l'on considère une surface F non réglée de S_3, l'image sur la quadrique de Klein des faisceaux tangents à F est une congruence quadratique K dont l'élément linéaire projectif de Segre est égal (peut-être au signe près) à l'élément linéaire projectif de Fubini de F. Dans cette représentation, aux lignes de Darboux de F correspondent les réglées principales de K, et réciproquement. *M. Decuyper* (Lille).

Upadhyay, M. D. Some properties of ϕ-congruences. Ganita **4**, 51–60 (1953).

Etant donnée une congruence rectiligne, si une autre congruence rectiligne de même surface de référence est telle que les rayons des deux congruences issus des mêmes points de la surface de référence se coupent sous un angle constant Φ, cette dernière congruence est dite par l'auteur "congruence Φ" vis-à-vis de la première. Prenant comme surface de référence le lieu des lignes de striction des surfaces réglées de la congruence donnée, l'auteur cherche les expressions des formes quadratiques de Kummer et de Sannia de la congruence ϕ associée, et en déduit les valeurs de la distance du point central d'une surface réglée issue d'un rayon de la congruence à la surface de référence choisie, et du paramètre de distribution de la surface réglée en question relatif au rayon envisagé. Il rattache aux résultats obtenus diverses propriétés des congruences Φ relatives, soit aux congruences Φ normales, soit à la représentation sphérique d'une congruence Φ, soit au comportement d'une congruence Φ, vis-à-vis de la surface de référence adoptée.

P. Vincensini (Marseille).

Mishra, R. S. On the congruences of lines. Ganita **2**, 83–88 (1951).

y^α $(\alpha=1, 2, 3)$ représentant le point courant de la surface de référence d'une congruence rectiligne dont les rayons dépendent des deux paramètres x^i $(i=1, 2)$, on connait les expressions de la quantité $p_i=\lambda^\alpha_{,i}y^\alpha_{,i}$, et celles [Behari et Mishra, Proc. Nat. Inst. Sci. India **15**, 85–92 (1949); ces Rev. **11**, 53] de $v^i=N^\alpha_{,i}\lambda^\alpha_{,i}$ où N^α est le vecteur unitaire normal de la surface de référénce. Dans la note actuelle l'auteur recherche l'expression de $r^i=\lambda^\alpha_{,i}N^\alpha_{,i}$ dans l'espace euclidien ordinaire. Il en déduit quelques nouvelles expressions des quantités p_i, et les étend au cas d'un $(m-n)$uple de congruences de courbes issues des points d'un sous-espace à n dimensions d'un espace riemannien à m dimensions.

P. Vincensini (Marseille).

Mishra, Ratan Shanker, and Shri Krishna. On the congruence of Ribaucour. Ganita 4, 39–49 (1953).

Les auteurs étudient, par les méthodes du calcul tensoriel, les congruences (de Ribaucour) obtenues en menant par un point quelconque d'une surface S_1 (de référence) la parallèle au point correspondant d'une surface S correspondant à S_1 avec orthogonalité des éléments linéaires, dite surface directrice de la congruence. Ils complètent les résultats antérieurement obtenus sur cette question par la considération de ce qu'ils appellent la congruence réciproque d'une congruence de Ribaucour donnée, c'est-à-dire, la congruence obtenue en intervertissant les rôles de S et de S_1. Le couple formé par deux congruences réciproques jouit de propriétés intéressantes que les auteurs étudient en détail, et qui mettent en jeu les fonctions caractéristiques de Bianchi pour les deux congruences, les surfaces associées [Eisenhart, A treatise on the differential geometry of curves and surfaces, Ginn, Boston, 1909] à leurs directrices, de même que les éléments d'aire correspondants autour des points homologues sur les surfaces de référence directrices et associées et sur les représentations sphériques des deux premières surfaces, ainsi que les positions des foyers et les distances focales sur l'un quelconque des rayons des congruences envisagées. *P. Vincensini* (Marseille).

Bhattacharya, P. B., and Ram Behari. Fundamental equations of condition to be satisfied by the coefficients of Sannia's quadratic forms. Ganita 4, 13–17 (1953).

Les auteurs retrouvent sous forme tensorielle les conditions auxquelles satisfont les coefficients des formes quadratiques différentielles de Kummer d'une congruence rectiligne, puis recherchent les conditions auxquelles doivent satisfaire les coefficients du système des deux formes quadratiques différentielles que Sannia a substitué au système de Kummer. *P. Vincensini* (Marseille).

Süray, Saffet. Sur des congruences de droites associées aux surfaces harmoniques. Communications Fac. Sci. Univ. Ankara. Sér. A. 4, 40–53 (1952). (Turkish summary)

P. Vincensini [Bull. Sci. Math. (2) 68, 60–72 (1944); ces Rev. 7, 78] a introduit les congruences (H) obtenues en projetant orthogonalement un point quelconque Q d'une surface harmonique $S[z=z(x, y)]$ sur le plan xOy, et en menant, par la projection m, la parallèle à la normale au point correspondant de S. Une telle congruence est à surface moyenne plane, et par suite du type de Ribaucour. Simonart et Alardin [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 602–613 (1949); ces Rev. 11, 396] ont défini un autre type de congruences de Ribaucour [les congruences (P)], jouissant de propriétés analogues aux congruences (H), admettant pour surfaces génératrices un paraboloïde de révolution d'axe parallèle à Oz, et obtenues en appliquant la construction donnant les congruences (H) à la perpendiculaire commune à S et à la surface harmonique conjuguée $S'[z'=z'(x, y)]$ en deux points homologues (situés sur une même parallèle à Oz). L'auteur a obtenu un certain nombre de résultats nouveaux relatifs aux congruences (H) et (P), et il en expose quelques uns dans le présent article. Il commence par étendre une propriété importante des congruences (H) aux congruences (P) en montrant que, si m est le point défini plus haut, et si $(m \to M)$ est une transformation conforme directe du plane xOy, la parallèle menée par M au rayon correspondant de (P) engendre une nouvelle congruence (P). Il déduit de cette propriété que l'intersection des plans tangents à deux surfaces harmoniques conjuguées en deux points homologues est une congruence (P); puis indique une construction géométrique rattachant, par l'intermédiaire d'une transformation conforme inverse du plan xOy, à toute congruence (H) une congruence (P) et inversement. Les congruences (H) restent telles si l'on fait tourner, d'un angle déterminé dans un sens déterminé, leurs différents rayons autour des parallèles à Oz issues des points m correspondants; l'auteur montre que cette propriété subsiste pour les congruences (P). Il établit ensuite que l'on obtient encore une congruence (P) en remplaçant les deux surfaces conjuguées qui interviennent dans la définition d'une telle congruence par deux surfaces harmoniques associées quelconques, puis que l'intersection des plans tangents à deux surfaces harmoniques associées en deux points homologues décrit une congruence (P), tout comme lorsque l'on considère deux surfaces harmoniques simplement conjuguées. *P. Vincensini* (Marseille).

***Finikov, S. P. System of congruences W with functional arbitrariness.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 169–174. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

Several systems of W-congruences are described, among them those named after Bianchi, and those obtained with the aid of a stratifiable pair of curves. Special attention is paid to the system given by the infinitesimal transformation of the reference tetrahedron $dA_i=\omega_i^k A_k$, $i, k=1, 2, 3, 4$, where

$$\omega_2^4=\omega_1^3, \quad \omega_4^3=\omega_2^1, \quad \omega_1^1-\omega_2^2-\omega_3^3+\omega_4^4=0,$$
$$\omega_3^1+\omega_4^2+a_3(\omega_1^3+\omega_2^4)=0,$$
$$\omega_3^2+a_1(\omega_1^3+\omega_2^4)+a_3\omega_1^2+a_3\omega_1^4=0,$$
$$\omega_4^1+a_4(\omega_1^3+\omega_2^4)-a_3\omega_2^1+a_3\omega_2^3=0,$$

and the a_1, a_2, a_3, a_4 satisfy four partial differential equations of the first order which follow from the equations of structure. The system W is determined by four arbitrary functions of one variable. The system of differential equations allows a one-dimensional characteristic manifold $\omega_1^3+\omega_2^4=0$, $\omega_1^2=0$, $\omega_2^1=0$, $\omega_1^4=0$, $\omega_3^3=0$. Here the edges A_1A_3 and A_2A_4 of the tetrahedron of reference slide on fixed lines. The system also allows a solution $a_2=0$, $a_3=0$ with three arbitrary functions of one variable. Here there is a characteristic manifold of three dimensions determined by $\omega_1^3+\omega_2^4=0$, $\omega_1^2=0$, $\omega_2^1=0$. *D. J. Struik* (Cambridge, Mass.).

Frazer, Lowell K. One-parameter families of linear line complexes. Tensor (N.S.) 2, 143–161 (1952).

The paper is an outline of a differential geometry of the one-parameter families of linear line complexes in euclidean space as a generalization of the differential geometry of ruled surfaces. The first section is concerned with the minimal linear imbedding sets. A linear set ${}_nL$ is defined by $n+1$ linearly independent complexes. The set of complexes $\mathbf{c}=\mathbf{c}(s)$ is called a ${}_nC$ if it is contained in a ${}_nL$ but in no ${}_{n-1}L(\mathbf{c}(s)=\{c^1(s), \cdots, c^n(s)\})$. Necessary and sufficient conditions for $\mathbf{c}(s)$ to be a ${}_nC$ are given. It follows a discussion of conjugate polars with respect to the complex $\mathbf{c}$ and a section regarding non-euclidean geometries of the sets ${}_nL$. Congruences having contact of first order with the set of congruences $\mathbf{c}=\mathbf{c}(s)$ (at the complex $\mathbf{c}(s)$) are called tangential congruences or congruences of accumulation at $\mathbf{c}(s)$. By putting $\mathbf{c}(s)=\mathbf{p}_1$, $\dot{\mathbf{c}}(s)=\mathbf{p}_2$, $\mathbf{p}_3\times\mathbf{p}_1=\mathbf{p}_3\times\mathbf{p}_2=0$, the author

gets relations between $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ and the derivatives $\dot{\mathfrak{p}}_1, \dot{\mathfrak{p}}_2, \dot{\mathfrak{p}}_3$. The scalar $K_1=\mathfrak{p}_3\times\mathfrak{p}_2$ is identically zero, if and only if the set $\mathfrak{c}=\mathfrak{c}(s)$ is a ${}_1C$. Another theorem gives what might be called the natural equation of the set $\mathfrak{c}=\mathfrak{c}(s)$. In the case of the metric geometry we have the invariant $\mathfrak{c}\cdot\mathfrak{c}$ and consequently the quantity known as the "Parameter" $P_s=\frac{1}{2}(\mathfrak{c}\times\mathfrak{c})(\mathfrak{c}\cdot\mathfrak{c})^{-1}$. If the parameter P_s is constant, then the locus of axes of $\mathfrak{c}(s)$ is a regulus, if and only if the set ${}_nC$ is a quadratic ${}_1C$. Furthermore the sets ${}_3C$ and ${}_nC$, $n\leq 5$, are discussed. In particular, the intrinsic equation of the ${}_4C$ is given. *M. Pinl* (Dacca).

Strubecker, Karl. Über die Flächen, deren Asymptotenlinien ein Quasi-Rückungsnetz bilden. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 103–110 (1953).

Sauer determined those surfaces

$$(U_1(u)+V_1(v),\ U_2(u)+V_2(v),\ z(u,v))$$

for which the parameter curves are the asymptotic lines [same S.-B. **1949**, 1–12 (1950); these Rev. **12**, 532]. The author makes various comments. 1) These surfaces have been discussed before. A footnote gives references. 2) They permit parametric representations which are free from integrals [Darboux, Leçons sur la théorie générale des surfaces, vol. 3, Gauthier-Villars, Paris, 1894, pp. 273–274]. 3) They can be dealt with by means of the author's differential geometry of isotropic space, i.e., of 3-space with the metric $ds^2=dx^2+dy^2$ [Math. Z. **47**, 743–777 (1942); these Rev. **7**, 530]. In this geometry, they are identical with the surfaces of constant relative curvature $K=z_{xx}z_{yy}-z^2_{xy}\neq 0$. If $K<0$, the asymptotic lines are real. If $K>0$, the representation 2) involves one single analytic function. 4) Many special cases are discussed. *P. Scherk.*

Luzin, N. N. On a regular solution of the problem of deformation of a surface with preservation of a conjugate net. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 83–91 (1953). (Russian)

Sretenskiĭ, L. N. Remarks on a posthumous work of N. N. Luzin on integration of the equations of deformation of surfaces with preservation of a conjugate net. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 75–82 (1953). (Russian)

Luzin's paper was found after his death and is incomplete. It is one gigantic calculation practically without comments. The paper of Sretenskiĭ gives the history of the problem and supplies the purpose of Luzin's calculation. Deformation with preservation of a conjugate net (a literal translation of the Russian is "bending on a principal base", a term whose origin is unknown to the reviewer) is an isometric deformation of a surface under which a certain conjugate net always stays a conjugate net. There are five earlier papers of Luzin on these deformations [C. R. (Doklady) Acad. Sci. URSS (N.S.) **19**, 23–27, 231–236 (1938); Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1939**, no. 2, 81–106; no. 7, 115–132, no. 10, 65–84], which deal with the existence and non-existence of surfaces with such deformations. The present paper contains contributions toward the following two problems: to find necessary and sufficient conditions 1) on the coefficients of the first fundamental form and 2) on the coefficients of the first and the second fundamental forms for a surface to be realized as a surface which possesses deformations with preservation of a conjugate net.

H. Busemann (Los Angeles, Calif.).

***Calapso, Renato. Sulle reti derivate dalle reti *O*.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 287–300. Casa Editrice Perrella, Roma, 1953.

Etant donné un réseau (ou système conjugué) (P) dont on désigne par u et v les deux familles de courbes, on dit (Guichard) qu'une congruence est harmonique au réseau si ses rayons sont dans les plans tangents de ce dernier, et si les foyers du rayon situé dans le plan tangent en P sont respectivement situés sur les deux tangentes aux courbes du réseau issues de P. L'intersection des rayons situés dans le plan tangent en P, de deux congruences harmoniques à (P), décrit un réseau (P') dit dérivé de (P), (P) étant l'anti-dérivé de (P').

L'auteur fait une belle application des formules de la théorie des déterminants orthogonaux de Guichard, en étudiant le problème suivant: Comment peut-on caractériser, dans l'espace euclidien à n dimensions, les réseaux (P) orthogonaux (réseaux O) tels que, parmi leurs dérivés, figure un réseau (P') qui soit lui-même (O), la distance (PP') des points homologues restant constante. Après avoir défini la courbure d'un réseau (O) et donné son expression, il commence par rechercher les réseaux, qu'il appelle (M), transformés de Combescure de (O), et tels que si K est la courbure de (O) et K' celle de (M), on ait $1/K+1/K'=C$, C étant une constante que, moyennant une homothétie on peut supposer égale à ± 1. Il passe ensuite au problème énoncé plus haut de la transformation, par le procédé de dérivation précédemment rappelé, d'un réseau (O) en un autre, la distance des points homologues étant constante. Le résultat est très élégant: Les réseaux (O) susceptibles de la transformation indiquée sont les réseaux (M). L'examen du cas où la transformation est involutive montre que ce cas caractérise l'espace à trois dimensions, les réseaux (O) donnant lieu à l'involution étant alors les réseaux de courbure des surfaces pseudosphériques. *P. Vincensini.*

Grotemeyer, K. P. Über die Verbiegung konvexer Flächen mit einer Randkurve, die Eigenschattengrenze ist. Math. Z. **58**, 272–280 (1953).

A surface homeomorphic to a disk of class C''' with positive Gauss curvature is bounded by a shadow boundary if its tangent planes along the boundary curve are parallel to a fixed line. A simple proof is given for a recent result of Rembs [Math. Z. **56**, 271–279 (1952); these Rev. **14**, 901] that such a disk is rigid under infinitesimal rigid transformations for which the boundary remains a shadow boundary. The same is proved for finite deformations. The first result is also true when only the length of the spherical image of the boundary remains stationary.

H. Busemann (Los Angeles, Calif.).

Jonas, Hans. Bestimmung von Flächenklassen auf Grund geforderter Biegungseigenschaften. Math. Nachr. **9**, 307–320 (1953).

A congruence of rays experiences a bending in the Beltrami sense if the rays, considered as emerging from a surface, are kept rigidly attached to the surface element while the surface undergoes an isometric deformation. The author here discusses classes of surfaces for which certain properties are preserved by all such bendings. One of the major problems considered is the following. Let a congruence of rays issuing from a surface S be a normal congruence, and let T be any one of the normal surfaces of this congruence. Determine the surfaces S for which, under all possible bendings,

the asymptotic lines of S correspond always to a conjugate net on the corresponding orthogonal surface T.

S. B. Jackson (College Park, Md.).

Backes, F. Nouvelles recherches sur l'applicabilité projective des surfaces. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **26**, no. 9, 32 pp. (1952).

Ce travail constitue une extension d'un mémoire antérieur [mêmes Mém. **24**, no. 9 (1950); ces Rev. **12**, 281]. Dans une première partie, on considère une correspondance ponctuelle entre deux surfaces (N_1), (N_2) qui conserve les lignes asymptotiques et telle que les developpables de la congruence engendrée par N_1N_2 découpent sur (N_1) et (N_2) des réseaux conjugués (u, v). Si ces surfaces sont projectivement applicables l'auteur montre que le birapport des points N_1, N_2 et des foyers F_1, F_2 de N_1N_2 est le quotient d'une fonction de u par une fonction de v, et dans le cas où ce birapport se réduit à une constante, la congruence engendrée par N_1N_2 est W de même que celle engendrée par la droite N_3N_4 d'intersection des plans tangents en N_1 et N_2. Dans une seconde partie, on étudie diverses réciproques et dans une troisième partie, on établit l'existence de tels couples de surfaces (N_1), (N_2) projectivement applicables, telles que les développables de la congruence N_1N_2 tracent sur (N_1) et (N_2) des réseaux conjugués et que le birapport $(N_1N_2F_1F_2)$ soit constant. Le mémoire se termine par l'étude de quelques autres propriétés de la figure. La méthode utilisée est celle du tétraèdre mobile, introduite par Demoulin en 1909 et dont E. Cartan a tiré un si grand parti.

M. Decuyper (Lille).

Vaccaro, Giuseppe. Sull'isometria di calotte appartenenti a due varietà. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 404–407 (1953).

Consider two varieties V_m and $\bar{V}_m$ which can be imbedded in Euclidean S_{m+n}. Let u^i and $\bar{u}^i$ be coordinate systems on the varieties with origins 0 and $\bar{0}$ respectively, and ds^2 and $d\bar{s}^2$ the respective metrics. It is proved that a relation

$$\bar{u}^i - u^i + \frac{1}{3!}c^i_{hkl}u^hu^ku^l + \cdots$$

always exists such that $ds^2 - d\bar{s}^2 = 0$ to within terms of the second degree in u^i. Moreover, if such a relation exists such that $ds^2 - d\bar{s}^2$ is zero to within terms of the third degree in u^i, it is necessary and sufficient that the curvatures of the two varieties be equal at 0 and $\bar{0}$ respectively.

C. B. Allendoerfer (Seattle, Wash.).

***Dalla Volta, Vittorio. Sulle calotte deformabili di ipersuperficie in uno spazio euclideo.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 313–316. Casa Editrice Perrella, Roma, 1953.

It is well known that a hypersurface in a Euclidean space R is in general rigid; if it can be deformed it must have a definite projective structure (F. Schur, L. Bianchi, U. Sbrana, E. Bompiani). The question can be examined locally, with respect to a cap of a certain order (a point on the hypersurface and its neighborhood of that order). For a cap of the third order the following theorem holds: If such a cap can be deformed it belongs to an algebraic hypersurface of order 3, V_n^3, having a linear space S_{n-2} through the point; at all points of S_{n-2} the V_n^3 has the same tangent hyperplane π; the section V' of V_n^3 with π is such that the tangent spaces of V' at points of S_{n-2} are (not ∞^{n-2}, but) the pairs of an involution. Other remarkable projective results for caps of the fourth order are given.

E. Bompiani (Rome).

***Dalla Volta, Vittorio. Geometria differenziale in grande.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 113–124. Casa Editrice Perrella, Roma, 1953.

An expository account of classical and recent results in global differential geometry due to H. Hopf, Chern, Allendoerfer, Bochner, and others. *G. A. Hedlund.*

Pan, T. K. The spherical curvature of a hypersurface in Euclidean space. Pacific J. Math. **3**, 461–466 (1953).

The notion of spherical curvature used is that appearing in Levi-Cività [Absolute differential calculus, new ed., Blackie, London-Glasgow, 1946, pp. 258–261]. The author in this note applies results which he has obtained on the curvature of vector fields [Amer. J. Math. **74**, 955–966 (1952); these Rev. **14**, 406] to give new interpretations both of the spherical curvature as introduced by Levi-Cività, and of the Gaussian curvature. *E. T. Davies.*

Sen, R. N. On a type of vector space. Proc. Nat. Inst. Sci. India **19**, 475–486 (1953).

Various theorems concerning parallel fields in Riemann spaces due to Walker [Quart. J. Math., Oxford Ser. **20**, 135–145 (1949); these Rev. **11**, 460] and Ruse [ibid. **20**, 218–234 (1949); Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 78–92 (1950); these Rev. **11**, 461; **12**, 358] are given an algebraic background by the author. He studies abstract finite-dimensional vector spaces over a field which satisfy appropriate axioms. By introducing a certain notion of invariance of subspaces, the writer shows how his results apply to the theory of parallel fields of Walker and Ruse.

A. Fialkow (Brooklyn, N. Y.).

Walker, A. G. The fibring of Riemannian manifolds. Proc. London Math. Soc. (3) **3**, 1–19 (1953).

If a Riemannian manifold, M^n, with positive definite metric admits a field of parallel r-planes, it is known that M is locally the Riemannian product of two Riemannian manifolds of dimensions r and $n-r$ respectively. It has been stated by T. Y. Thomas [Monatsh. Math. Phys. **47**, 388–418 (1939)] that if M is complete it is a Riemannian product manifold (in the large). This paper shows that Thomas's result is not necessarily correct unless M is simply-connected and satisfies Postulates 1 and 2 below. It further discusses the wider issue of the fibring of M in the usual sense.

On the hypothesis that M is complete and that a field of parallel r-planes exists on M, it is shown that there exists a complete, geodesic submanifold $R(x)$ of r dimensions tangent to the r-planes and passing through the arbitrary point x. There is also an $(n-r)$-dimensional, complete, geodesic submanifold $S(x)$ tangent to the dual field of parallel $(n-r)$-planes. M is now restricted by Postulate 1: "There is in M an $S(x)$, say S_0, such that every point of S_0 belongs to a subset of S_0 which is open in the topology of S_0 and does not meet any $R(x)$ in more than one point." On this basis is proved the theorem: "M is a fibre-bundle with $R(x)$ as fibres and whose base space is covered isometrically by S_0." It is then shown that every $S(x)$ has the properties required of S_0.

The paper continues with a discussion of the structure group, G, of the fibre bundle. If it is assumed (Postulate 2)

that the only automorphism belonging to the structure group and having an invariant point is the identity, it follows that any two submanifolds $S(x)$ and $S(x')$ are homeomorphic and isometric. If G is the identity, M is reducible to the product of two manifolds. In general M cannot be so reduced, but can be covered isometrically with a reducible manifold. *C. B. Allendoerfer.*

*Varga, Ottó. **Normalkoordinaten in allgemeinen differentialgeometrischen Räumen und ihre Verwendung zur Bestimmung sämtlicher Differentialinvarianten.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 131–162. Akadémiai Kiadó, Budapest, 1952. (Hungarian and German. Russian summary)

Consider a manifold of line-elements (u, v) with an affinor $C^{\nu}_{\alpha\beta}(u, v)$ and an affine connection $\Gamma^{*\gamma}_{\alpha\beta}(u, v)$. The author introduces normal coordinates and normal affinors which are generalizations of those defined by T. Y. Thomas and Veblen for an affinely-connected manifold. He then proves a generalization of Thomas's Replacement Theorem: The components of any affinor-differential invariant are functions of the covariant derivatives of $C^{\nu}_{\alpha\beta}$ (and its partial derivatives with respect to v's), the derivatives with respect to v's of $\Gamma^{*\gamma}_{\alpha\beta}$, the normal affinors, and the results of multiplying the last two groups of these affinors by v^{ρ}.

C. B. Allendoerfer (Seattle, Wash.).

Seki, Setsuya. On the curvatura integra in a Riemannian manifold. Comment. Math. Univ. St. Paul. 1, 41–50 (1953).

The author gives a proof of the generalized Gauss-Bonnet theorem similar to that of Chern. The innovation consists of a new way of handling the calculations in the neighborhood of a singularity of the imposed vector field.

C. B. Allendoerfer (Seattle, Wash.).

Calabi, Eugenio. Isometric imbedding of complex manifolds. Ann. of Math. (2) 58, 1–23 (1953).

The purpose of the paper under review is to give the necessary and sufficient conditions for the isometric imbedding, in the small as well as in the large, of a complex manifold in a unitary space of finite or infinite dimension, and also those for the isometric imbedding of a complex manifold in the Hermitian manifolds of Fubini and Study.

After a short introduction the author introduces in chapter 2 the so-called diastatic function. Let M^k be a k-dimensional complex manifold with a positive definite Hermite-Kaehler metric $ds^2 = g_{\alpha\beta^*}(z, \bar{z})dz^{\alpha}d\bar{z}^{\beta}$ whose components are real analytic functions of the coordinates. The Hermite-Kaehler metric is characterized by the existence of a real-valued analytic scalar $\Phi(z, \bar{z})$ such that $g_{\alpha\beta^*}(z, \bar{z}) = \partial^2\Phi(z, \bar{z})/\partial z^{\alpha}\partial\bar{z}^{\beta}$. Two functions Φ' and Φ satisfying this relation are related by $\Phi' - \Phi = \phi(z) + \overline{\phi(z)}$, where $\phi(z)$ is a complex analytic function of z^{α}. Now, let P and Q be two arbitrary points of a coordinate neighborhood (z) with coordinates $z_P{}^{\alpha}$ and $z_Q{}^{\alpha}$ respectively; then the functional element

$$(*)\quad D(P, Q) = \Phi(z_P, \bar{z}_P) + \Phi(z_Q, \bar{z}_Q) - \Phi(z_P, \bar{z}_Q) - \Phi(z_Q, \bar{z}_P)$$

is called the diastasis or diastatic function of P and Q. The name "diastasis" is justified by the fact that, if $\rho(P, Q)$ is the geodesic distance between P and Q and $D(P, Q)$ the functional element of the diastasis defined by (*), then $D(P, Q) = (\rho(P, Q))^2 + O((\rho(P, Q))^4)$.

In chapter 3, the author considers the problem of the isometric complex analytic imbedding of complex manifolds in the following three types of flat spaces: (a) the unitary spaces C^N, or spaces with the coordinate system $(x) = (x^1, x^2, \cdots, x^N)$ and the metric $ds^2 = \sum_{\sigma=1}^{N} |dx^{\sigma}|^2$; (b) the countable infinite-dimensional unitary or complex Hilbert space C^{∞} of points with coordinates

$$(x) = (x^1, x^2, \cdots, x^{\sigma}, \cdots)$$

such that $\sum_{\sigma=1}^{\infty} |x^{\sigma}|^2 < \infty$, with the metric $ds^2 = \sum_{\sigma=1}^{\infty} |dx^{\sigma}|^2$; (c) the indefinite Hilbert space E, defined as the space of points whose coordinates are $(x^1, x^{-1}, x^2, x^{-2}, \cdots, x^{\sigma}, x^{-\sigma}, \cdots)$ such that $\sum_{\sigma}' |x^{\sigma}|^2 < \infty$, with the indefinite Hermitian form $ds^2 = \sum_{\sigma}' (\operatorname{sgn}\sigma)\,|dx^{\sigma}|^2$, $\sum_{\sigma}'$ denoting the infinite sum for σ ranging over all non-zero integers.

Now, the author's first theorem on complex analytic imbedding is: A complex manifold M^k with a metric can be locally imbedded isometrically and complex analytically in the indefinite Hilbert space E if and only if the metric is an analytic Kaehler metric. The author proceeds next to the case where the ambient space is a positive definite unitary space and proves first the local rigidity: Let M^k be a Kaehler manifold and assume that a given arbitrarily small region G of M^k can be isometrically and complex analytically imbedded in a unitary space C^N of dimension N ($N \leqq \infty$), so that it does not lie in any proper complex analytic linear subspace of C^N. Then N is uniquely determined by the Kaehler metric in G, and the imbedding is determined up to within the group of motions in C^N.

The author next tries to characterize the conditions for a complex manifold to satisfy the assumptions of the above theorem and he obtains: Let M^k be a Kaehler manifold with analytic metric, P_0 a point of M^k, (z) a complex coordinate system with origin at P_0. Let G be the neighborhood of P_0 in which the power series expansion of the diastasis $D(P_0, P)$ in terms of $(z_P, \bar{z}_P)$ converges. Then, for any given positive integer $N \leqq \infty$, there exists a neighborhood of P_0 that can be isometrically imbedded in C^N if and only if the coefficient matrix $B_{J,K}$ of the power series expansion $D(P_0, P) = \sum_{J,K=1}^{\infty} B_{J,K}(z_P)^{(m)_J}(\bar{z}_P)^{(m)_K}$ is positive semi-definite and of rank at most N. Furthermore, if the condition is satisfied, at least all of G can be isometrically imbedded in C^N. Here we are using the abbreviated notation:

$$\sum_{J,K=1}^{\infty} B_{J,K}(z_P)^{(m)_J}(\bar{z}_P)^{(m)_K} = \sum_{\substack{m_1, \cdots, m_k;\ n_1, \cdots, n_k \geq 0 \\ \sum_{\alpha} m_{\alpha} \geq 1,\ \sum_{\alpha} n_{\alpha} \geq 1}} B_{m_1, \cdots, m_k; n_1, \cdots, n_k} \prod_{\alpha=1}^{k} [(z_P^{\alpha})^{m_{\alpha}}(\bar{z}_P^{\alpha})^{n_{\alpha}}].$$

Now, a Kaehler manifold M^k with analytic metric is called resolvable of rank N at a point P_0 if, for a coordinate system having P_0 as origin, the local power series expansion of the diastasis $D(P_0, P) = \sum_{J,K=1}^{\infty} B_{J,K}(z_P)^{(m)_J}(\bar{z}_P)^{(m)_K}$ has the property that the coefficient matrix $B_{J,K}$ is positive semi-definite and of rank N. After this definition, the author proves the global character of resolvability: If a Kaehler manifold M^k, whose metric is analytic, is resolvable of rank N at one point P_0, then it is resolvable of rank N at each of its points. He proves moreover that: A simply connected Kaehler manifold M^k can be globally imbedded in C^N if and only if the metric is analytic and M^k is resolvable of rank at most N. If M^k is resolvable of rank $N' \leqq N$, then the image of M^k in C^N spans linearly a subspace of C^N of dimension N'.

Now, the author's main theorem is: A complex manifold M^k with any metric admits a global isometric, complex

analytic imbedding in C^N if and only if the following conditions are fulfilled: (a) the metric is an analytic Kaehler metric; (b) the Kaehler manifold is resolvable of rank at most N at any point; (c) for each point P the analytic extension of the diastasis $D(P, Q)$ with respect to Q over M^k is single-valued. The imbedding is, in addition, topological (i.e., without self-intersections) if and only if (d) $D(P, Q)=0$ only for $P=Q$.

The author moreover proves a general theorem on the relation between the canonical coordinates of a Kaehler manifold and those of a complex analytic submanifold: Let M^k be a finite-dimensional analytic submanifold of an N-dimensional ($N \leqq \infty$) Kaehler manifold S^N, whose metric is analytic. Then the induced metric on M^k is an analytic Kaehler metric and if $(z^1, z^2, \cdots, z^k)$ is a canonical system of coordinates in M^k with respect to a point P_0 of M^k, then there exists a canonical coordinate system (x) in S^N with respect to P_0 such that the imbedding equations are

$$x^\sigma = z^\sigma \qquad (\sigma = 1, 2, \cdots, k)$$
$$x^\sigma = f^\sigma(z) = \sum_{J=k+1}^{\infty} b^\sigma_J (z)^{(m)}{}_J \qquad (\sigma = k+1, \cdots, N),$$

where the range of indices in the summation implies that the imbedding functions $f^\sigma(z)$ contain no term of degree less than 2 in (z).

In chapter 4, the author discusses the problem of the isometric and complex analytic imbedding of complex manifolds in Fubini spaces, obtaining theorems corresponding to those mentioned above. In the last chapter, the author gives the conditions that a Fubini space can be globally imbedded isometrically into another Fubini space.

K. Yano (Rome).

Cossu, Aldo. Alcune osservazioni sul confronto tra connessioni affini e metriche riemanniane. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 29–35 (1953).

Let a variety X_n be endowed with an asymmetric connection L^h_{ik} and a metric g_{hk}. The connection is called "Euclidean with respect to the metric" if the length of every vector (computed using g_{hk}) is invariant under parallel displacement according to L^h_{ik}. The torsion vector corresponding to a 2-direction $(dx^h, \delta x^h)$ is defined as $S^i_{hk} dx^h \delta x^k$ where S^i_{hk} is the skew-symmetric part of L^h_{ik}. It is proved that a connection which is Euclidean with respect to the metric has the metric geodesics as its autoparallel curves if and only if the torsion vector of every 2-direction is normal to this direction. A related theorem is proved for the case when L^h_{ik} is not Euclidean.

There is determined the most general form of a connection such that the length of an arbitrary vector is unchanged under displacements orthogonal to a given vector. An important special case is a "metric connection" which preserves angles under parallel displacement. Necessary and sufficient conditions are given that a metric connection should have the metric geodesics as its autoparallel curves.

C. B. Allendoerfer (Seattle, Wash.).

Ślebodziński, Władysław. Géométrie textile et les espaces à connexion affine. Rozprawy Mat. **3**, 34 pp. (1953).

The author presents a new method of studying the invariants of sets of curves or hypersurfaces in an n-dimensional space. He shows that the nets consisting of n or $n+1$ families of curves or of $n+1$ families of hypersurfaces can be put into one-to-one correspondence with the paths of a variety with a particular affine connection (P). The invariants of the nets are then studied by known methods which apply to such a variety.

The first chapter considers conditions on a symmetric affine connection such that the paths can be separated into ∞^{n-1} congruences whose directions are in homographic correspondence at different points. From this connection a second connection (P) is derived which is non-symmetric and which results in absolute parallelism of vectors. A solution is given to the problem of the equivalence of two varieties with the connection (P). In Chapter II the invariants of a net of three families of curves in the plane are obtained. In Chapter III the same problem is treated for families of $n+1$ curves or hypersurfaces in a space of n-dimensions. The solution depends upon the properties of the connection (P). Chapter IV considers the situation for n families of curves in an n-space.

C. B. Allendoerfer.

Deicke, Arno. Über die Darstellung von Finsler Räumen durch nichtholonome Mannigfaltigkeiten in Riemannschen Räumen. Arch. Math. **4**, 234–238 (1953).

A necessary and sufficient condition is obtained for the possibility of locally imbedding a Finsler F_n in a Riemannian R_{2n-1} subject to the conditions: (1) The points of R_{2n-1} are in one-to-one correspondence with the line elements of F_n; (2) at every point of R_{2n-1} there is an n-direction such that the local metric and the Cartan connection of F_n can be obtained by projecting the corresponding quantities of R_{2n-1} upon this n-direction. When the imbedding does exist, it turns out that these n-directions are actually tangent to submanifolds on R_{2n-1}, so that the "non-holonomic" submanifolds mentioned in the title are in fact holonomic.

C. B. Allendoerfer (Seattle, Wash.).

Tonooka, Keinosuke. Geometrical treatment of an $(n-1)$-ple integral. Tensor (N.S.) **2**, 108–122 (1952).

This paper is very similar to a previous one by the author [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **12**, 43–72 (1952); these Rev. **14**, 1017] in which he developed an intrinsic theory of the manifold of surface elements of higher order. The word intrinsic is used to indicate that quantities are considered with respect to both coordinate and parameter transformations. The difference lies in the fact that whereas in the previous paper two neighboring surface elements $(x^i, p^i_{\alpha(1)}, \cdots, p^i_{\alpha(m)})$ and $(x^i+dx^i, p^i_{\alpha(1)}+dp^i_{\alpha(1)}, \cdots)$ were regarded as referring to the same value of the parameters u, in this paper the differential dp refers to a variation in the u as well as in the p.

E. T. Davies (Southampton).

Flanders, Harley. Development of an extended exterior differential calculus. Trans. Amer. Math. Soc. **75**, 311–326 (1953).

It is a well-known fact that an affine connection on a differentiable manifold gives rise to covariant differentiations of tensor fields. The classical approach makes use of the natural frames relative to local coordinates and works with the components of tensor fields, thus giving the impression that this branch of differential geometry is a venture through a maze of indices. The author gives a mechanism which shows that this is not necessarily so. Even for the more general case of a connection in a fiber bundle with a Lie structural group, indications of such an approach have been given [H. Cartan, Colloque de topologie, Bruxelles, 1950, Thone, Liège, 1951, pp. 15–27; these Rev. **13**, 107; S. Chern, Topics in differential geometry, mimeographed notes, Princeton, 1951]. However, this is the first systematic and detailed account for affine connections. The author considers the space $\mathfrak{T}_q{}^p$ of q-forms with p-vectors as coefficients (of class ∞), so that $\mathfrak{T}_0{}^1$ is the space of all vector fields

and $\mathfrak{T}_1^0$ that of all linear differential forms. An affine connection is defined as an operator d of $\mathfrak{T}_0^1$ into $\mathfrak{T}_1^1$, having the properties: 1) $d(\mathbf{v}+\mathbf{w})=d\mathbf{v}+d\mathbf{w}$; 2) $d(f\mathbf{v})=df\mathbf{v}+fd\mathbf{v}$, where f is a scalar function and $\mathbf{v}$, $\mathbf{w}$ are vector fields. This operator induces an operator d of $\mathfrak{T}_q^p$ into $\mathfrak{T}^p{}_{q+1}$, the covariant differentiation. Curvature and torsion are defined in essentially the same way as by E. Cartan, but without indices. The most original parts of the paper are in §9.10, in which formulas are derived for the repeated applications of the operator d. We quote the following simple one: $d(d\mathbf{v})^r=rd^2\mathbf{v}(d\mathbf{v})^{r-1}$. Such formulas allow the author to derive in a more elegant way formulas given by the reviewer in his proof of the Gauss-Bonnet formula [Ann. of Math. (2) **45**, 747–752 (1944); **46**, 674–684 (1945); these Rev. **6**, 106; **7**, 328]. Finally, it is shown that the present definition of the affine connection coincides with the classical formulation.

S. Chern (Chicago, Ill.).

Bassaly, W. A. Applications of group theory to the study of the symmetry properties of tensors. Proc. Math. Phys. Soc. Egypt **4** (1952), no. 4, 105–116 (1953).

It is well known that the second rank tensor $F_{i_1i_2}$ $(i_1, i_2=1, \cdots, n)$ can be completely reduced into its completely symmetric and completely antisymmetric parts according to

$$F_{i_1i_2}=\tfrac{1}{2}(F_{i_1i_2}+F_{i_2i_1})+\tfrac{1}{2}(F_{i_1i_2}-F_{i_2i_1}).$$

The problem to be solved is to obtain similar reductions for tensors of higher ranks having maximum symmetry properties. In this paper a method is used to reduce third and fourth rank tensors which are of interest to physics. The symmetry properties of the Riemann-Christoffel tensor which plays an important part in Einstein's theory of relativity are shown to be maximum. (Author's summary.)

A. Nijenhuis (Princeton, N. J.).

Bade, W. L., and Jehle, Herbert. An introduction to spinors. Rev. Modern Physics **25**, 714–728 (1953).

The authors give a fairly extensive review of the theory of two component spinors and their application to the Dirac and the Maxwell equations. The theory of such spinor fields over a general Riemannian space is also discussed. The latter discussion includes a theory of covariant differentiation.

A. H. Taub (Urbana, Ill.).

Ôneda, Sadao, and Umezawa, Hiroomi. On the families of spinor fields. Progress Theoret. Physics **9**, 685–687 (1953).

The authors define a family of spinor fields as sets of spinor fields of different kinds such that those of one kind anti-commute with those of another kind. An attempt is then made to classify spinor fields corresponding to known particles into various families. Three alternative classifications are discussed.

A. H. Taub (Urbana, Ill.).

NUMERICAL AND GRAPHICAL METHODS

***von Sanden, H. Praktische Mathematik. Eine Einführung mit besonderer Berücksichtigung von Statistik und Ausgleichsrechnung.** 3te Aufl. B. G. Teubner Verlagsgesellschaft, Leipzig, 1953. 128 pp. DM 3.20.

The second edition of this book was reviewed in these Rev. **13**, 584. In this edition the author has considerably enlarged Chapter V on the method of least squares. The concept of mean error has been more precisely presented by mean of the concept formulated in statistics, a new section has been added on the mean error of a proportion and also a new section on the approximation of empirical functions by rational entire functions.

W. E. Milne (Corvallis, Ore.).

***Hrenov, L. S. Semiznačnye tablicy trigonometričeskih funkcil. [Seven-place tables of trigonometric functions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 415 pp. 33.75 rubles.

These tables are intended to give the astronomer a good seven-place grip on the natural values of the trigonometric functions. One third of the volume is devoted to the cotangents and cosecants of angles less than ten degrees. Here the interval is one second of arc. This is followed by a table of all six functions at the interval of ten seconds. First differences and proportional parts are given in the main table. Presumably these tables are the result of collating the tables of Gifford, Andoyer and Peters. The large page (29.1×22 cm.) allows adequate room for the main table. The printing is a little irregular but the paper is of better quality than usual. At least seven and sometimes eight significant figures are given. There are nine minor conversion tables mainly for astronomical computers.

D. H. Lehmer (Los Angeles, Calif.).

***Tables des fonctions de Legendre associées. Fonction associée de première espèce $P_n^m(\cos\theta)$. Premier fascicule: [n: $-0, 5(0, 1)10$], [m: $0(1)5$], [θ: $0(1)90°$].** Editions de La Revue d'Optique, Paris, 1952. xxi+292 pp. 3000 francs.

Legendre functions whose degree is not an integer are encountered in potential and wave problems for regions bounded by a cone, in potential problems for regions bounded by a hyperboloid of revolution, and in other similar problems. Apart from the case when the degree is half of an integer, no systematic tabulation seems to exist, and these tables represent a pioneering effort in a field which grows in importance.

The present volume contains: a Preface by J. Coulomb; an Introduction by L. Robin giving the definition

$$P_n^m(\cos\theta)=(-\sin\theta)^m\frac{d^mP_n(\cos\theta)}{d(\cos\theta)^m}$$

(m a non-negative integer, n arbitrary) and the basic properties of P_n^m; an account by P. LeGall of the method of computation, checks, accuracy; a table giving the number of terms in the power series used to compute the functions for $\theta=0°(1°)90°$ (45 terms when $\theta=90°$); a diagram facilitating the use of recurrence formulas; the level curves of $P_n^0(\cos\theta)=\text{const.}, \cdots, P_n^5(\cos\theta)=\text{const.}$ in the region $0\leqq n\leqq 0, 0°\leqq\theta\leqq 90°$ of the n, θ plane; a detailed page index; and 290 pp. of numerical tables of $P_n^m(\cos\theta)$ for $m=0(1)5$, $n=-.5(.1)10$, $\theta=0°(1°)90°$.

A second volume will extend the tabulation to $\theta=180°$.

A. Erdélyi (Pasadena, Calif.).

***Salzer, Herbert E. Tables of coefficients for the numerical calculation of Laplace transforms.** National Bureau of Standards Applied Mathematics Series, No. 30. U. S. Government Printing Office, Washington 25, D. C. ii+36 pp. $0.25.

This pamphlet is concerned with the approximate numerical evaluation of integrals of the form

$$(*) \qquad \int_0^\infty e^{-pt} f(t)\,dt$$

as a sum of the form $\sum_{i=0}^{n-1} A_i^{(n)}(p) f(i)$. The formula can be made exact, by suitable choice of the A_i, if $f(i)$ is a polynomial of degree not exceeding $n-1$; otherwise its accuracy depends on the accuracy with which $f(t)$ may be approximated by such a polynomial. The tables are designed for use when $f(t)$ is replaced by a Lagrange interpolation polynomial of degree $n-1$ for $t=0(1)n-1$. Examples of the use of the tables and of the determination of error estimates are given.

The tables themselves are as follows. Schedule A: Lagrange interpolation coefficients. This gives $L_i^{(n)}(t)$ defined by the approximate formula

$$f(t) \doteq \sum_{i=0}^{n-1} L_i^{(n)}(t) f(i)/(n-1)!$$

as polynomials in t for $n=2(1)11$, $i=0(1)n-1$. Schedule B: Laplace transforms of interpolation coefficients. This gives coefficients $B_i^{(n)}(p)$ as polynomials in p, for the same ranges of n and i, where $p^n B_i^{(n)}(p)(n-1)!$ is given by (*) when $f(t)$ is the Lagrange coefficient $L_i^{(n)}(t)/(n-1)!$. The polynomials in the second schedule may be obtained from those in the first by replacing t^r by $r!\,p^{n-r-1}$.

Table I gives 8 decimal (or 8 figure) numerical values of $A_i^{(n)}(p)$ for $n=2(1)11$, $i=0(1)n-1$, and for

$$\begin{aligned} p&=0.1(0.1)n-1 \quad \text{to} \quad n=7\\ p&=0.2(0.2)n-1 \quad \text{to} \quad n=9\\ p&=\ 1(1)n-1 \quad \text{to} \quad n=11. \end{aligned}$$

Table II gives 9 figure values of $n!/p^{n+1}$ for $n=0(1)10$, $p=0.1(0.1)10$.

It is a pity to have to comment that the arrangement of the tables leaves very much to be desired.

J. C. P. Miller (Cambridge, England).

Iverson, K. E. The zeros of the partial sums of e^z. Math. Tables and Other Aids to Computation **7**, 165–168 (1953).

This paper contains 12 decimal values of the solutions of the algebraic equations $S_n(z)=\sum_0^n z^k/k!=0$ for $n=2(1)23$. All roots are complex except for one real root whenever n is odd. Several properties of $S_n(z)$ and its zeros are listed, and a diagram of the plotted zeros is given.

J. C. P. Miller (Cambridge, England).

Slater, L. J. On the evaluation of the confluent hypergeometric function. Proc. Cambridge Philos. Soc. **49**, 612–622 (1953).

In this paper the author provides an 8S table of ${}_1F_1(a;b;x)$ for $a=-1(.1)1$, $b=.1(.1)1$, $x=1(1)10$, and an asymptotic expansion for larger x. For the asymptotic expansion, converging factors are determined by a method developed by J. C. P. Miller [same Proc. **48**, 243–254 (1952); these Rev. **13**, 836]. *A. Erdélyi* (Pasadena, Calif.).

Fetzer, Viktor. Die praktische Berechnung der elliptischen Funktion "sn" von Jakobi unter besonderer Berücksichtigung der Bestimmung der Cauer-Parameter für Filter mit Betriebsdämpfungsverhalten. Arch. Elektr. Übertragung **7**, 393–401 (1953).

The author provides formulas for the numerical computation of $\mathrm{sn}(pK/q)$ where p and q are integers and $0<p<q\leqq 12$. If q is odd, $\mathrm{sn}(K/q)$ is computed by means of the Fourier series; the case of an even q is reduced to that of an odd q by using the half-angle formulas. For $p>1$, $\mathrm{sn}(pK/q)$ is computed from $\mathrm{sn}(K/q)$ by means of the addition and duplication theorems. *A. Erdélyi* (Pasadena, Calif.).

Yuasa, Tosiko, et Laberrigue-Frolow, Jeanne. Remarques sur les valeurs numériques de la fonction de Fermi. J. Phys. Radium (8) **14**, 95–99 (1953).

For a Coulomb field e^2Z/R, Fermi's function is

$$F(Z,W)=(2p\tilde{R})^{2s-2}2(s+1)e^{\pi t}|\Gamma(s+it)/\Gamma(2s+1)|^2$$

where $\alpha=1/137$, $s^2=1-\alpha^2Z^2$, $p^2=W^2-1$, $t=\alpha ZW/p$, and $\tilde{R}=\hbar/(m_0c)$. Since tables for the gamma function in the complex domain are inadequate, physicists often use approximate expressions. The present authors computed $|\Gamma(s+it)|^2$ to five significant figures and give graphs of $f(\pm Z,W)$ for $Z=1$, 5(5)90 and $1.1\leqq W/(m_0c^2)\leqq 6$. (They also refer to a table which does not appear to have been printed with the paper.) They compare their values with those obtained from the approximate formulas of Bethe and Bacher, and Nordheim and Yost. *A. Erdélyi.*

Tomlinson, R. C. A revised index of mathematical tables for compressible flow. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda **2691** (1949), 12 pp. (1953).

Dupuy, Michel. Résumé de la thèse de M. Dupuy sur l'interpolation complexe et ses applications en géodésie et cartographie. Bull. Géodésique **1953**, 257–264 (1953). (German, Spanish and Italian summaries)

Schöbe, Waldemar. Angenäherte Summation und Rekursion mittels der Lubbockschen Formel. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 3, 3–13 (1952).

The author advocates rewriting Lubbock's formula [Steffensen, Interpolation, Williams and Wilkins, Baltimore, Md., 1927] in Cotes form with remainder term and provides numerical illustrations. *H. L. Seal.*

Guest, P. G. On the standard errors in the fitting of polynomials to unequally spaced observations. Australian J. Physics **6**, 131–154 (1953).

An approximate procedure is presented to compute orthogonal polynomial values $T_j(x)$ when the independent variable x is unequally spaced. Given a set of n such x's, designated as $x(\epsilon)$, where

$$\epsilon=-\tfrac{1}{2}(n-1),\ -\tfrac{1}{2}(n-3),\ \cdots,\ \tfrac{1}{2}(n-1).$$

$x(\epsilon)$ is first approximated by

$$x(\epsilon)=\sum_{j=0}^{3} k_j T_j(\epsilon),$$

where $k_j=\sum_\epsilon T_j(\epsilon)x(\epsilon)/\sum_\epsilon T_j^2(\epsilon)$. Approximate procedures are presented to compute the k_j. Two functions of the k's specify the departure from uniform spacing: $\kappa_2=n\varphi k_2$, for symmetry; $\kappa_3=n^2\varphi k_3/2$, for concentration the scalar $\varphi=(k_1+n^2k_3/10)^{-1}$.

Tables for given values of κ_2 and κ_3 are presented to simplify the computations of the constants in the usual recursion formulas used to compute successive T_j's; tables are also presented to facilitate the computation of approximate standard errors of fitted values. Several examples are presented. *R. L. Anderson* (Raleigh, N. C.).

***Forsythe, George E. Tentative classification of methods and bibliography on solving systems of linear equations.** Simultaneous linear equations and the determination of eigenvalues, pp. 1–28. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

***Fröberg, Carl-Erik. Solutions of linear systems of equations on a relay machine.** Simultaneous linear equations and the determination of eigenvalues, pp. 39–42. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

***Frazer, R. A. Some problems in aerodynamics and structural engineering related to eigenvalues.** Simultaneous linear equations and the determination of eigenvalues, pp. 65–74. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

***Petrie, George W., III. Matrix inversion and solution of simultaneous linear algebraic equations with the IBM 604 Electronic Calculating Punch.** Simultaneous linear equations and the determination of eigenvalues, pp. 107–112. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

***Todd, John. Experiments on the inversion of a 16×16 matrix.** Simultaneous linear equations and the determination of eigenvalues, pp. 113–115. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

***Sherman, Jack. Computations relating to inverse matrices.** Simultaneous linear equations and the determination of eigenvalues, pp. 123–124. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

Ostrowski, A. On over and under relaxation in the theory of the cyclic single step iteration. Math. Tables and Other Aids to Computation 7, 152–159 (1953).

The cycle single-step iteration for the solution of a system of linear equations is frequently described as Seidel's iteration process. In order to accelerate convergence it has been suggested that the process be modified by the use of "incomplete relaxation"; the ith variable is chosen not to liquidate the ith residue R_i exactly, but to replace it by $(1-q_i)R_i$. If $0<q_i\leqq 1$ we have under-relaxation, if $1<q_i\leqq 2$ over-relaxation. The author points out that this device does not always speed up the relaxation procedure. However, the cyclic single step iteration (if converging) applied to a real 2×2 matrix (symmetric or not) is shown to benefit from a suitable incomplete relaxation always. Convergence can even be forced in this way for many cases of unsymmetric matrices. A numerical example is included.

O. Taussky-Todd (Washington, D. C.).

***Semon, Warren L. Characteristic numbers and their use in the decomposition of switching functions.** Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 273–280. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

For the purpose of the design of electronic switching circuits, the resolution of a binary function $f(x_1, \cdots, x_n)$ of n binary variables into the form $F(\varphi, \psi)$, where F is a binary function and φ is a binary function of some subset of $\{x_1, x_2, x_3, x_4\}$ and ψ is a binary function of the complementary subset, is of interest. With the aid of the Mark I Automatic Sequence Controlled Calculator a table of results for $n=4$ has been computed. *S. Sherman.*

Davidenko, D. F. On approximate solution of systems of nonlinear equations. Ukrain. Mat. Žurnal 5, 196–206 (1953). (Russian)

Elaboration, with application to two numerical examples, of a method already described in Doklady Akad. Nauk SSSR (N.S.) 88, 601–602 (1953) [these Rev. 14, 906].

A. S. Householder (Oak Ridge, Tenn.).

***Krylov, V. I. Interpolation of the highest order of accuracy in the problem of indefinite integration.** Trudy Mat. Inst. Steklov., v. 38, pp. 97–145. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

This paper gives a careful and scholarly extension of Gauss's method of numerical integration so as to include formulas of the more general type

$$y(x_0)=\sum_{i=1}^{r+s} A_i y(x_i)+\sum_{i=r+1}^{r+s+l} B_i y'(x_i)$$

where the right hand member contains y alone at the points $x_1, \cdots, x_r$, contains both y and y' at the points $x_{r+1}, \cdots, x_{r+s}$, and y' alone at the points $x_{r+s+1}, \cdots, x_{r+s+l}$. The author establishes theorems which determine the highest possible order of formulas of this type when certain ones of the x's are preassigned and the others are determined, as in Gauss' formulas, by the zeros of an appropriate polynomial. Further theorems determine this polynomial for any given case.

Having established the general theory, the writer proceeds to detailed examination of a set of special cases for which $x_0, x_1, \cdots, x_{r+s}$ are given equally spaced values with step interval h and the remaining x's are determined to maximize the order of the formulas. These cases are as follows. Case 1. $r=1$, $s=0$. (This gives Gauss' formulas, which are well known.) Case 2. $r=0$, $s=1$. (This case gives Markov's formulas, less well known.) Case 3. $r=1$, $s=1$. Case 4. $r=0$, $s=2$. As far as the reviewer is aware, cases 3 and 4 are new.

The author gives the values of the x's and the coefficients A_i and B_i to seven decimal places for the following:

Case 2. $r=0$, $s=1$, $l=1, 2, 3, 4, 5, 6$.
Case 3. $r=1$, $s=1$, $l=1, 2, 3, 4$.
Case 4. $r=0$, $s=2$, $l=1, 2, 3, 4$.

Finally a rigorous bound for the error is obtained and is worked out in detail for a number of cases, including most of those listed above. The error term is of the form $Ch^{n+1}y^{(n+1)}(\xi)$, where C is a numerical constant dependent on the particular formula but independent of the function

$y(x)$, n is the order of the formula, and ξ lies between the least and the largest x occurring in the formula. [The analysis of the error is substantially that used by Milne, Numerical calculus . . . , Princeton, 1949, pp. 108–116; these Rev. 10, 483.]

W. E. Milne (Corvallis, Ore.).

Grebenyuk, D. G. Construction of formulas for approximate calculation of double integrals in a region (D) representing a circle $x^2+y^2=k^2$. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 9, 29–59 (1952). (Russian)

The author derives quadrature formulas for the approximate evaluation of the integral $I=\int_{(D)}f(x, y)dxdy$, where (D) is the region bounded by the circle $x^2+y^2=k^2$. The formulas are of the type $I=\sum A_{ij}f(x_i, y_j)$ where (x_i, y_j) are points in the circle. The principal results are: 1) a four-point formula using the points $(\pm k/2, \pm k/2)$; 2) a six-point formula using the two points $(0, \pm k/2)$ and the four points $(\pm k/\sqrt{2}, \pm k/\sqrt{2})$; 3) a nine-point formula using $(0, 0)$, $(0, \pm k/\sqrt{2})$, $(\pm k/\sqrt{2}, 0)$ and $(\pm k/\sqrt{2}, \pm k/\sqrt{2})$ (as printed this formula is wrong, one term having been overlooked); 4) a sixteen-point formula using

$$(\pm[(3\pm\sqrt{5})/8]^{1/2}, \pm[(3\pm\sqrt{5})/8]^{1/2}),$$

all sixteen choices of sign being made. The error is not examined.

W. E. Milne (Corvallis, Ore.).

Sala, Ilmari. Numerische Lösung von linearen homogenen Eigenwertaufgaben zweiter Ordnung durch Mittelwertmethoden. Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 13, 9 pp. (1950).

The author is concerned with approximate numerical methods of determining the characteristic values λ of a linear homogeneous boundary value problem of a 2nd order equation of the type $y''+\lambda g(x)y(x)=0$. Using the familiar Green's function $\Gamma(x, t)$ of the equation the problem is reduced to the integral equation $y(x)=\lambda c\int\Gamma(x, t)g(t)y(t)dt$. Representing the integral by a weighted sum of integrands at n equidistant arguments t_i and using a Taylor expansion for the product $g(t)y(t)$, an approximate determinantal equation for λ is obtained. As an example the characteristic values for the equation $y''+\lambda x^{-m}y=0$ with boundary conditions $x=b$, $y=d$, $y'=0$; $x=b-l$; $y=0$ are computed.

H. O. Hartley (Ames, Iowa).

Karpilovskaya, È. B. On convergence of an interpolation method for ordinary differential equations. Uspehi Matem. Nauk (N.S.) 8, no. 3(55), 111–118 (1953). (Russian)

The author applies general theorems of L. V. Kantorovich concerning approximate solutions of equations containing a linear operator to the case of a non-homogeneous linear differential equation of order $2m$ with variable coefficients $p_i(t)$ and homogeneous two-point boundary conditions. She assumes that, for all i, $|p_i^{(r)}(t_1)-p_i^{(r)}(t_2)|<L|t_1-t_2|^\alpha$, where r is an integer and $0<\alpha\leq 1$. Then using the collocation method of approximating the solution at n points inside the interval she finds an error of order $O(\ln^2 n/n^{r+\alpha})$ when the collocation points are zeros of the nth degree Čebyšev polynomial and $O(1/n^{r+\alpha-1})$ when they are zeros of the nth degree Legendre polynomial. *W. E. Milne.*

***Collatz, L. Détermination numérique de solutions périodiques dans le cas d'oscillations non linéaires. Zur numerischen Bestimmung periodischer Lösungen bei nichtlinearen Schwingungen.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 195–205; discussion, p. 206, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and German)

The author exhibits several procedures for finding periodic solutions of non-linear differential equations with periodic boundary conditions. One such method uses relaxation to determine the unknown coefficients in a finite trigonometric sum assumed to satisfy the equation. If the period is unknown it also is one of the quantities to be determined. Another method uses finite differences and is applicable in the case where the differential equation can be put in the form $L(x)=N(x^{(m)}, t)$, where L is a linear differential operator with constant coefficients and N is a non-linear function of some definite derivative of x, containing the independent variable t. The methods are illustrated by worked examples.

W. E. Milne (Corvallis, Ore.).

Mikeladze, M. Š. Numerical solution of a system of differential equations. Application of the method to the computation of rotating shells. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 382–386 (1953). (Russian)

The author begins with a system of ordinary differential equations, each containing derivatives of second order only. With Taylor's theorem, such a system may be written

$$y_\nu''(\phi)=F_\nu(\phi)+\sum_{\mu=1}^{n}a_{\nu\mu}\left[y_\mu(\alpha)+(\phi-\alpha)y_\mu'(\alpha)+\int_\alpha^\phi(\phi-t)y_\mu''(t)dt\right] \quad (\nu=1, 2, \cdots n).$$

He then finds values of $y_\nu''(\phi)$ at equally spaced abscissae, $\phi=kh$, in the interval $\alpha\leq\phi\leq\beta$ by replacing the integral shown with its trapezoidal approximation, obtaining a recurrence relation

$$y_{\nu k}''=F_{\nu k}+\sum_{\mu=1}^{n}a_{\nu\mu}^{(k)}\left[y_\mu(\alpha)+khy_\mu'(\alpha)+\frac{kh^2}{2}y_{\mu 0}''+h^2\sum_{p=1}^{k-1}(k-p)y_{\mu p}''\right].$$

Here $y_{\nu k}''$, $F_{\nu k}$, $a_{\nu\mu}^{(k)}$ refer to $y_\nu''(kh)$, $F_\nu(kh)$, $a_{\nu\mu}(kh)$, respectively. As an application of this method, the author writes equations for displacements in a rotating shell of variable thickness, and transforms them to his required form. Numerical values pertaining to a certain spherical shell are chosen, and four steps of the calculation are carried out.

R. E. Gaskell (Seattle, Wash.).

Kirkby, S., and Nonweiler, T. The numerical solution of certain differential equations occurring in Crocco's theory of the laminar boundary layer. Coll. Aeronaut. Cranfield. Rep. no. 74, 9 pp. (1953).

A numerical method is described for the solution of certain differential equations which result from the application of Crocco's transformation to the laminar boundary-layer equations appropriate to high supersonic Mach numbers (i.e., at hypersonic speeds). The solution is obtained by continuous application of a rapidly convergent relaxation process to a pair of simultaneous differential equations, for which one of the boundary conditions is a first derivative. The Prandtl number occurs as a parameter. (Authors' summary.)

H. Polachek (Carderock, Md.).

Karlqvist, Olle. Numerical solution of elliptic difference equations by matrix methods. Tellus 4 (1952), 374–384 (1953).

A non-iterative method for solving boundary-value problems of elliptic difference equations is developed that is based on the simple structure of the triangular matrices obtained by applying the well-known elimination procedure of Gauss to the matrix of the difference equation. It is proved that the procedure is numerically stable, i.e., there is no tendency to error-growth. The method seems to be particularly suited to such problems as weather forecasting on high-speed computing machines where one has to solve the same equations (e.g., Poisson's or Helmholtz's) a great number of times with the same net but different right member and boundary values. The number of stored constants as well as the computational work in the application of the method is only a fractional part of that required when using the inverse matrix (Green's function). Poisson's equation for a rectangular region is given a close study, but the method can be applied to irregular regions and to equations of higher order. (Author's summary.)

H. Polachek (Carderock, Md.).

v. Mohrenstein, A. Numerische Auflösung der Schrödinger-Gleichung. Z. Physik 134, 488–503 (1953).

The author describes (without giving fully worked examples) a numerical method of solving the Schroedinger equation $H\psi = E\psi$ for a number of problems concerning the state of He and H_2 atoms. The method is a development of that suggested by A. A. Frost [J. Chem. Phys. 10, 240–245 (1942)] and consists briefly of (i) representing ψ in the form $\sum \alpha_j \phi_j$ using a suitably chosen set of complete space functions ϕ_j and unknown coefficients α_j, (ii) evaluating the values $\epsilon_i = H\psi_i/\psi_i$ at a 'grid' of N points and determining the α_j by minimising $\delta = N^{-1}\sum \psi_i^2(\epsilon_i - \bar{\epsilon})^2$ where the summation is taken over the 'grid', (iii) iteratively examining the convergence of this procedure. The author's modification of the process is concerned with an improved and computationally more economical method of estimating and modifying the weight functions ψ_i^2 in the least square fit of the α_j.

H. O. Hartley (Ames, Iowa).

Rényi, Alfréd. Remarques concernant un traité de P. Gombás et R. Gáspár. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 1 (1952), 393–397 (1953). (Hungarian. Russian and French summaries)

Le travail contient quelques remarques mathématiques en connexion avec le travail, Acta Phys. Acad. Sci. Hungaricae 1, 66–74 (1951) [ces Rev. 13, 993].

Résumé de l'auteur.

Sekiya, Tsuyoshi, and Tsutsui, Saburo. On the approximate solution of the boundary-value problem for the plane biharmonic equation. J. Osaka Inst. Sci. Tech. Part I. 3, 43–67 (1951). (Esperanto)

The authors consider a finite-difference form of the two-dimensional biharmonic equation $\Delta^2 u = 0$ for both regular hexagonal lattice nets and equilateral triangular nets. The corresponding difference equations are solved by an approximate method for a particular boundary-value problem. Finally, two sets of integral equations, formally derived from plane potential theory, are stated for the equilibrium problem for a clamped plate. *C. R. DePrima.*

Fichera, Galtano. Results of recent experiments in the analysis of periods carried out in the Istituto Nazionale per le Applicazioni del Calcolo. Simultaneous linear equations and the determination of eigenvalues, pp. 125–126. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

Jánossy, Lajos. Searching for periodicities. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 7–25 (1953). (Hungarian)

The author discusses various iterative methods for the detection of periodicities in observational data.

E. Lukacs (Washington, D. C.).

Couffignal, Louis. Méthodes et limites de la Cybernétique. Revista Acad. Ci. Madrid 47, 63–82 (1953).

The author compares both analogue and electronic digital machines with the behaviour of human beings and animals. Some interesting analogies with respect to the system of nerves are pointed out. As a whole it is the author's opinion, that only units of machines and units of natural beings should be compared with one another and that the definition of "cybernetics" should be taken wide enough to include the collaboration of different disciplines of the sciences of nature. The reviewer enjoyed the author's delicate approach to a delicate question. *H. Bückner* (Holloman, N. M.).

Nemes, T. Logical machine for recognizing class and causal relations genetically. Acta Tech. Acad. Sci. Hungar. 7, 3–17 (1953). (Russian, French, and German summaries)

The machine described is based on a modification of the Euler diagrams as follows: The regions A, B, C, etc., are horizontal rows of dots; the inclusion of A in B is indicated by having a B dot in every column which contains an A dot; the overlapping of A by B by having at least one column which has both an A dot and a B dot; etc. The machine is a device for sensing and recording such coincidences. By its aid one can test the validity of syllogisms and similar arguments, including some properties of propositional algebra. By certain modifications one can test coincidences with time-log, so that there is a possibility of use in detecting possibilities of causal connection. *H. B. Curry.*

Frey, Tamás. Les principes de fonctionnement des diverses sortes de planimètres. (Les machines de l'Institut de Mathématique Appliquée.) Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 1 (1952), 253–294 (1953). (Hungarian. Russian and French summaries)

L'article donne une description détaillée du principe de fonctionnement et de la construction des planimètres polaires, linéaires et radiaux, ainsi que des planimètres de Stieltjes. Il discute systématiquement et d'une manière détaillée la théorie des planimètres fonctionnels et intégrimètres fonctionnels convenables pour évaluer les intégrales ayant les formes

$$\int_a^b f[y(x)]dx \quad \text{et} \quad \int_a^b f[y(x)]g(x)dx.$$

Résumé de l'auteur.

Dainelli, Dino, e Aparo, Enzo. Considerazioni sulle più recenti macchine calcolatrici elettroniche ed il loro impiego. Ricerca Sci. 23, 1528–1549 (1953).

Bird, R. Computing machines, input and output. Electronic Engrg. **25**, 407–410 (1953).

Townsend, R. Serial digital adders for a variable radix of notation. Electronic Engrg. **25**, 410–416 (1953).

van der Poel, W. L. Dead programmes for a magnetic drum automatic computer. Appl. Sci. Research B. **3**, 190–198 (1953).

Brown, J. A. C., Houthakker, H. S., and Prais, S. J. Electronic computation in economic statistics. J. Amer. Statist. Assoc. **48**, 414–428 (1953).

Ramsayer, K. A function calculator as applied to geodesy. Bull. Géodésique **1953**, 275–292 (1953). (German, Spanish, French and Italian summaries)

Clippinger, R. F., Dimsdale, B., and Levin, J. H. Automatic digital computers in industrial research. I. J. Soc. Indust. Appl. Math. **1**, 1–15 (1953).

ASTRONOMY

Jeffreys, Harold. The figures of rotating planets. Monthly Not. Roy. Astr. Soc. **113**, 97–105 (1953).

The theory of the figure of the Earth on the hydrostatic hypothesis, developed by Darwin and de Sitter, is applied to models for Jupiter and Saturn proposed by W. H. Ramsey and B. Miles [Ramsey, same Not. **111**, 427–447 (1951); Miles and Ramsey, ibid. **112**, 234–243 (1952)].

Author's summary.

Batson, Lewis. A simplification in orbit computation. Astr. J. **58**, 169–170 (1953).

The Laplacian method of orbit determination requires the evaluation of determinants of the third order. It was shown by K. Stumpff [Astr. Nachr. **243**, 317–336 (1931); **244**, 433–464 (1932)] that a form of solution may be obtained in which the determinants are of the second order. The modification of Stumpff's procedure presented here retains the advantage of second-order determinants and permits a convenient tabulation of the solution of the principal equations. Let ρ designate the geocentric, r the heliocentric distance of the object, R the distance of the earth from the sun, and $\bar{\rho}=\rho/R$, $\bar{r}=r/R$. The principal equations are obtained in the form $\bar{\rho}=\bar{A}[1-\bar{r}^{-3}]$, $\bar{r}^2=\bar{\rho}^2+\bar{B}\bar{\rho}+1$. A double entry table of the solution $\bar{\rho}$ as a function of $\bar{A}$ and $\bar{B}$ is given. The range of tabulation is appropriate for applications to minor planets. *D. Brouwer.*

Thüring, B. Analyse der Beziehungen zwischen Inertialsystem, Gravitation, Präzession und Eigenbewegungen der Fixsterne. Astr. Nachr. **281**, 49–58 (1953).

The basic thesis of this article is that an inertial system and the universal law of gravitation cannot both be determined uniquely from observed motions of planets or stars. The proof for the two-body problem, extended to any number of bodies, depends on the circumstance that the number of equations is smaller than the number of unknowns. On this ground the author considers illusory the attempts to determine an inertial system from the secular variations of the elements of planetary orbits. Similarly, the validity of defining an inertial system with the aid of stellar proper motions with reference to galactic rotation is dismissed because the law of force which determines the rotation of the galactic system is still incompletely known.

As an alternative the author proposes to determine an inertial system from proper motions of stars in a fundamental catalogue by the condition that the number of stars for which the proper motions in the inertial system are very small (say, numerically smaller than the mean error) shall be a maximum.

[The criterion is a suitable formulation of the determination of an inertial system from distant extragalactic nebulae, to which the author refers. A serious flaw in its application to galactic objects is that no allowance is made for the systematic effects of solar motion and galactic rotation. This renders the method one of very limited power and a step backward compared with solutions in which these systematic features are taken into account.

In extending the basic thesis from the two-body problem to that of any number of bodies the author ignores the crucial requirement that one single universal law of gravitation must account for all features of the motions of all bodies in the system.] *D. Brouwer.*

Masotti, Arnaldo. Osservazioni sui moti kepleriani. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 345–353 (1951).

The absolute values of the differences between the eccentric and mean anomalies, true and eccentric anomalies and true and mean anomalies (equation of the center) in elliptic or Keplerian motion have maxima for $r=a$, $r=b$ and $r=(ab)^{1/2}$, respectively. The following definition is introduced: mean of F with respect to $A=(2\pi)^{-1/2}\int_0^{2\pi}FdA$, in which F is a quantity depending upon the position of the planet and A one of the anomalies. Then the following properties of Keplerian motion are shown. 1) In the points for which the absolute value of the difference between the eccentric and mean anomalies reaches a maximum the kinetic energy is equal to its time-mean. 2) In the points where the absolute value of the difference between the true and eccentric anomalies reaches a maximum the kinetic energy is equal to its mean with respect to the eccentric anomalies. 3) In the points where the absolute value of the equation of the center reaches a maximum the central attraction is equal to its time-mean. 4) The time-mean of the kinetic energy depends only upon the major axis.

A. J. J. van Woerkom (New Haven, Conn.).

Masotti, Arnaldo. Sopra una generalizzazione dei moti centrali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 448–460 (1951).

In this paper pseudo-central motion is defined as motion for which the modulus of the area vector is constant as a generalization of central motion where the area vector itself is a constant. It is shown that for pseudo-central motion the acceleration can be resolved into two components, one in the direction of the radius vector and one in the direction of the instantaneous area vector. The velocity and acceleration components are also given with respect to a moving coordinate system for which the X-axis points in the direction of the radius vector and the Z-axis in the direction of the instantaneous area vector. The latter part of the paper deals with an elementary treatment of the determination of the

force when the path is given and the determination of the path when the force is given. *A. J. J. van Woerkom.*

Sokolov, Yu. D. On a case of integrability of the equations of symmetric motion of a system of three material points. Ukrain. Mat. Žurnal 3, 347–380 (1951). (Russian)

This paper is a continuation of an earlier one by the same author [same Žurnal 2, no. 3, 7–44 (1950); these Rev. 13, 996]. Consider a system of three particles P_i of masses m_i $(i=0, 1, 2)$, which attract or repel each other, the interaction between P_i and P_j $(i\neq j)$ having magnitude $m_im_j|f(r_{ij})|$ and representing an attraction or repulsion according as f is negative or positive. Further assume that $f(r)=Ar+B/r^3$, where A and $B\neq 0$ are arbitrary numbers. Let $m_1=m_2$ and let the initial conditions at $t=0$ be chosen in such a way that the triangle $P_2P_0P_1$ remains isosceles during the whole motion. It was shown that the only possible types of motion of this triangle are: (i) rotation about the axis parallel to its base through the center of inertia of the system, (ii) rotation about its axis of symmetry (altitude), and (iii) planar motion in which P_0 moves along a fixed straight line while the particles P_1 and P_2 describe trajectories which are symmetric with respect to this line. A complete discussion in terms of elliptic functions of these possible cases of the relative motion is given. Some of the results obtained generalize certain earlier results of P. V. Voronec [Universitetskiya Izvestiya, Kiev 47, nos. 1, 2, Čast' II, 180+iii pp. (1907)]. *E. Leimanis.*

Sitnikov, K. On the possibility of capture in the problem of three bodies. Doklady Akad. Nauk SSSR (N.S.) 87, 521–522 (1952). (Russian)

The author describes, without details of proof, a three-parameter family of solutions of the three-body problem, which illustrate "capture"; that is, as time t approaches $-\infty$, all mutual distances become infinite, whereas, as t approaches $+\infty$, the distance between two of the bodies remains bounded but the distance from either to the third becomes infinite.

W. Kaplan (Ann Arbor, Mich.).

Chandrasekhar, S., and Fermi, E. Problems of gravitational stability in the presence of a magnetic field. Astrophys. J. 118, 116–141 (1953).

This paper considers the gravitational stability of cosmical masses of infinite electrical conductivity situated in a magnetic field. In Section I the authors have extended the virial theorem to include the magnetic terms in the equations of motion, and show that when the magnetic energy exceeds numerically the gravitational potential energy, the configuration becomes dynamically unstable. In Section II the adiabatic radial pulsations of an infinite cylinder are considered along the axis of the magnetic field, and an expression for the period is obtained. Section III is devoted to an investigation of stability for transverse oscillations of an infinite cylinder of incompressible fluid in a uniform magnetic field acting in the direction of the axis. It is shown that the cylinder is unstable for all periodic deformations of the boundary whose wave lengths exceed a certain critical value depending on the strength of the field. It is found that this field has a stabilizing effect, both in increasing the wave length of maximum instability, and in prolonging the time needed for the instability to make itself felt. For a cylinder of radius 250 parsecs, and of mean density of 2×10^{-24} g/cm^3, a magnetic field in excess of 7×10^{-6} gauss effectively removes the instability. In Section IV it is shown that a fluid sphere with a uniform magnetic field inside, and a dipole field outside, is not a configuration of equilibrium, and will tend to become oblate by contracting in the direction of the field. Finally, in Section V the gravitational instability of an infinite homogeneous medium in the presence of a magnetic field is considered, and it is shown that Jeans' condition for the onset of instability is unaffected by the presence of the field. *Z. Kopal.*

Brahmachary, R. L. Sur la possibilité d'un nouveau modèle statique de cosmologie. I, II. Naturwissenschaften 40, 51, 313–314 (1953).

The author proposes a stationary cosmological model with a homogeneous distribution of electric charges. He studies the properties of this model in two special cases, when space is Euclidean, and when the density of charged matter is negligible. *A. Schild* (Pittsburgh, Pa.).

RELATIVITY

Vaidya, P. C. Spherically symmetric solutions in nonsymmetrical field theories. I. The skew symmetric tensor. Physical Rev. (2) 90, 695–698 (1953).

The author studies skew-symmetric space-time tensors g_{ik} which are to be spherically symmetric in 3-space. He obtains expressions which are more general than those of Papapetrou [Proc. Roy. Irish Acad. Sect. A. 52, 69–86 (1948); these Rev. 10, 580]. However, it seems to this reviewer that the author's use of the concept of spherical symmetry is unusual, and that he is in fact finding tensors which have axial symmetry of rotation about a fixed direction in space. *A. Schild* (Pittsburgh, Pa.).

Raychaudhuri, Amal Kumar. Reine Strahlungsfelder mit Zentralsymmetrie in der Allgemeinen Relativitätstheorie. Z. Physik 135, 225–231 (1953).

The author shows that there are no solutions of Einstein's gravitational equations which represent a pure electromagnetic radiation field and which are regular and spherically symmetric, although not necessarily static. He concludes that a spherically symmetric material body, such as a star, cannot radiate away all its mass. *A. Schild.*

Lenoir, Marcel. Une généralisation du théorème de Gauss. C. R. Acad. Sci. Paris 237, 384–385 (1953).

The author obtains for Einstein's unified theory the following generalization of Gauss's theorem. If S is a closed hypersurface enclosing a volume V in which ds^2 is regular, and if the field is stationary, then

$$\text{flux}_S H=\int\int\int\int_V \{g^{ik}R_{ik}+g^{ik}\bar{R}_{ik}\}\sqrt{(-g)}dx^1dx^2dx^3dx^4.$$

Here H is the space-vector defined by

$$H^h=-\{g^{h(\alpha)}\Gamma^{(\delta)}_{(\delta)(\alpha)}+g^{(\alpha)h}\Gamma^{(\delta)}_{(\alpha)(\delta)}\},$$

bracketed indices being "dead". The notation is Einstein's and the methods are based on those of Lichnerowicz [Problèmes globaux en mécanique relativiste, Hermann,

Paris, 1939, p. 65; these Rev. 1, 282], who obtained the corresponding theorem for general relativity.

H. S. Ruse (Leeds).

Lenoir, Marcel. Théorème de régularité dans la dernière théorie d'Einstein. C. R. Acad. Sci. Paris **237**, 424–425 (1953).

The author quotes a theorem of Lichnerowicz [Problèmes globaux en mécanique relativiste, Hermann, Paris, 1939, p. 43; these Rev. 1, 282] that every exterior space-time which is stationary and everywhere regular is Euclidean, and seeks a generalization of the theorem in the unified theory of Einstein. In the cases he considers he finds that the theorem does not in fact hold. *H. S. Ruse.*

Jordan, Pascual. Über Gravitationswellen in der erweiterten Gravitationstheorie. Elektrotech. Z. **74**, 144 (1953).

Die Einsteinschen Vakuumfeldgleichungen $G_{kl}=0$ gestatten die "Weylsche" Lösung $\eta_{kl}=\xi_{k|l}+\xi_{l|k}$ (ξ_k beliebige differenzierbare Funktionen, $\xi_{k|l}=\partial\xi_k/\partial x_l$). Da diese Lösungen durch Koordinatentransformation zum Verschwinden gebracht werden können, stellen sie keinen physikalischen Vorgang dar. Die Feldgleichungen lassen sich jedoch so modifieren, dass nunmehr ein Teil der η_{kl} "Weylsche Lösungen darstellen, andere unter ihnen aber, die jetzt noch zusätzlich die Wellengleichung $\Box\eta_{kl}=0$ befriedigen, physikalisch reale Wellen vorstellen, die nicht mehr durch Koordinatentransformationen zum Verschwinden gebracht werden können. Die Gravitationswellen der Einsteinschen Theorie lassen sich auch in die durch Verf. und andere erweiterte Gravitationstheorie mit variabler "Gravitationskonstanten" übernehmen. Die linearisierten Feldgleichungen dieser Theorie werden durch $\eta_{kl}=-\delta_{kl}\omega$ gelöst, wenn ω der Wellengleichung genügt. *M. Pinl* (Dacca).

Castoldi, Luigi. Sulla struttura formale della relatività e su una classe notevole di connessioni metriche di interesse relativistico. Atti Accad. Ligure **9** (1952), 5–14 (1953).

Starting with the equations

$$R^{ij}-\tfrac{1}{2}a^{ij}R=T^{ij}-E^{ij},$$

where a^{ij} is the contravariant fundamental tensor, R^{ij} is the Ricci tensor and T^{ij}, E^{ij} are the material and electromagnetic energy-tensors defined in the usual way in terms of the momentum 4-vector and electromagnetic 6-vector respectively, the author lays down a principle of "maximum and simplest determination" which requires that the field-equations should involve the smallest possible number of arbitrary elements, and that they should be as simple as possible in form. He uses this principle to deduce the fundamental equations of the gravitational and electromagnetic fields. The form of certain of the equations leads him to consider a metric connection L^k_{ij} defined by

$$L^k_{ij}=\left\{\begin{matrix}k\\ij\end{matrix}\right\}+q_iF^k_j$$

where q_i is a vector and F_{ij} a 6-vector. This he calls a degenerate connection (degenerate in the sense of not being the most general of its kind), and a special case of it a self-degenerate connection. He concludes that, for a universe containing only discrete charged material particles, the geometrical description may be made in terms of a self-degenerate connection. *H. S. Ruse* (Leeds).

Ingarden, R. S. Equations of motion and field equations in five-dimensional unified relativity theory. Doklady Akad. Nauk SSSR (N.S.) **88**, 773–776 (1953). (Russian)

The author considers a five-dimensional Riemannian space described by means of a coordinate system subject to certain conditions. The gravitational equations of the general relativity theory involving the energy-momentum density tensor of matter are generalized to five dimensions. However, the latter tensor is assumed to have a vanishing trace, so that a material particle has a zero rest-mass and moves along the "light cone" in the five-dimensional space. This leads the author to interpret the fifth coordinate as the "proper time" from the standpoint of four-dimensional relativity theory. However, the author fails to discuss various difficulties associated with this point of view, such as that connected with the possible dependence of the field variables on the fifth coordinate. *N. Rosen* (Haifa).

Zaycoff, Raschko. Verallgemeinerte Theorie der Gravitation. Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **1** (1950), 244–262 (1951). (Bulgarian. Russian and German summaries)

The author modifies the formalism of the general theory of relativity by introducing, in addition to the metric tensor, a gravitational four-vector. He does this because he considers that there is too large a discrepancy between the prediction of the original theory concerning the deflection of light passing close to the sun and observation. It is true that in the modified theory he is able to get good agreement with observation for all three of the crucial phenomena, but this is accomplished by adjusting various constants at his disposal. From the conceptual standpoint, having so many elements in the theory does not seem satisfactory.

N. Rosen (Haifa).

Kalitzin, Nikola. Elektromagnetismus und Gravitation. Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **2** (1951), 49–62 (1952). (Bulgarian. Russian and German summaries)

This paper is based on the earlier work of G. Nordström [Phys. Z. **15**, 504–506 (1914); cf. also H. C. Corben, Physical Rev. (2) **69**, 225–234 (1946); these Rev. **7**, 533], in which the Maxwell equations for the electromagnetic field are generalized to five dimensions, the quantities associated with the fifth dimension then serving to describe the gravitational field. It is assumed that there exist two kinds of particles, having opposite charges and represented by world lines in five dimensions with velocity vectors in opposite directions. A neutral particle is assumed to be made up of a pair of such particles.

It should be noted that while such an approach gives, in a suitable approximation, the Newtonian description of gravitation, it fails to give the more accurate description provided by the general theory of relativity. *N. Rosen.*

Kalitzin, Nikola St. Eine Verallgemeinerung der Gleichungen der Elektrodynamik. Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **2** (1951), 63–78 (1952). (Bulgarian. Russian and German summaries)

The generalization of the equations of electrodynamics involves using a space of six dimensions instead of four and a potential which is an antisymmetric tensor instead of a vector. The equations are interpreted in Minkowski four-space on the assumption that the field variables are independent of the coordinates associated with the other two

dimensions. The equations are interpreted as describing electromagnetic and gravitational fields. On the other hand, by assuming that the variables have a periodic exponential dependence on the two new coordinates, one gets equations of the Klein-Gordon or Yukawa type. However, the interpretation of the field variables is not clear since no satisfactory equations of motion are set up. *N. Rosen.*

Zaycoff, Raschko. Einheitliche Feldtheorie im sechsdimensionalen Raum. Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **2** (1951), 79–98 (1952). (Bulgarian. Russian and German summaries)

This work is similar to that of Kalitzin [see the preceding review], but has the advantage that the field equations are so chosen that it is possible to set up appropriate equations of motion. In addition to the electromagnetic and gravitational fields, a new field is present which is apparently connected with the spin properties of matter. The author is able to obtain the three crucial effects of the general relativity theory, including a value for the deflection of light by the sun that appears to be in better agreement with observation than that given by the latter; but this involves a certain amount of adjustment of the integration constants to fit the data. *N. Rosen* (Haifa).

Todeschini, Bartolomeo. Sul potenziale elettromagnetico nella teoria unitaria di Einstein. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 495–500 (1953).

The author first notes that, in general relativity, the electromagnetic field-equations

$$\epsilon^{klmn}F_{mn|k}=0, \quad F_{mn}{}^{|n}=J_m \tag{1}$$

(a vertical stroke denoting Levi-Cività covariant differentiation) lead to the equations

$$\phi^n{}_{m|n}+\phi^i R_{im}=-J_m \tag{2}$$

for the potential 4-vector ϕ_n defined by

$$F_{mn}=\phi_{n|m}-\phi_{m|n} \quad \text{and} \quad \phi_n{}^{|n}=0. \tag{3}$$

He then writes down the equations corresponding to (1) in the unified theory of Einstein, and obtains those corresponding to (2). He notes incidentally that, if F_{mn} is identified with the conjugate ${}^*g_{mn}=\frac{1}{2}\epsilon_{mnpq}g^{pq}$ of the skew part of the asymmetric fundamental tensor, then equations (3) express that skew part in terms of the potential 4-vector. *H. S. Ruse* (Leeds).

Scherrer, W. Metrisches Feld und vektorielles Materiefeld. II. Comment. Math. Helv. **27**, 157–164 (1953).

The previous paper of the same name [Comment. Math. Helv. **26**, 184–202 (1952); these Rev. **14**, 417] is generalized to permit the basic invariant W to contain, linearly, the second derivatives of the matter-field four-potential. *A. J. Coleman* (Toronto, Ont.).

Moon, Parry, and Spencer, Domina Eberle. Binary stars and the velocity of light. J. Opt. Soc. Amer. **43**, 635–641 (1953).

The authors consider two hypotheses with regard to the velocity of light: (a) it is constant relative to all observers (Einstein); (b) it is constant relative to the source (Ritz). They reject (a) as being contrary to all human experience and involving the abolition of ordinary ideas of space and time. They reject (b) also since it implies certain peculiarities in the apparent motions of distant binaries which have not been observed. They propose a third hypothesis (c), which is (b) with the modification that, while astronomical space remains Euclidean for material bodies, light is considered to travel in a Riemannian space of constant positive curvature; on the basis of astronomical data they assign the value $R=5$ light years to the radius. [The reviewer does not understand what the hypothesis (c) means.] *J. L. Synge* (Dublin).

MECHANICS

*Pérès, Joseph. **Mécanique générale.** Masson et Cie, Paris, 1953. iv+407 pp. Broché 2110 francs; cartonné toile 2545 francs.

In this book the traditional content of a French course on mechanics is presented in a compact style of writing (which sacrifices no clarity) with compact notation and great attention to detail. Vectors are used, indicated by a superposed bar. Much useless writing is saved by the concept of a torseur, written $[\mathfrak{T}]$, this being a system of sliding vectors, two torseurs being equal if their vector sums and moments are equal. Thus the fundamental law of mechanics (used in this form systematically throughout the book) reads $[\mathfrak{a}_a]=[\mathfrak{F}_a]$, the first term being the torseur of quantities of acceleration (the vectors mass×acceleration for the particles forming the system) and the second the torseur of external forces. The subscript a indicates an absolute frame of reference; moving frames (which are treated extensively) are taken care of by adding to $[\mathfrak{F}_a]$ the torseur of fictitious forces. The first three chapters cover principles, mechanics of perfect solids (rigid bodies), virtual work, and the theorem of energy. The author employs right-handed axes and the usual definition of a positive rotation; to the reviewer it seems a pity that he adheres to force vive and force function instead of using kinetic and potential energies, which terms are mentioned but not much employed in the book. Chap. IV is devoted to the differential equation $(dq/dt)^2=F(q)$ which occurs so often in mechanics. Chaps. V and VI deal with problems in the motion of a particle and a rigid body, including Foucault's pendulum and gyroscopic effects. Chap. VII gives Lagrange's equations and Appell's; it is indicative of the nature of the book that the integral of energy drops out as a particular case of Painlevé's integral, valid when T is quadratic but not homogeneous in the velocities. Chap. VIII deals with variational principles and Hamilton's equations; it includes an extremely neat transition from Hamilton's principle to the principle of least action. The last three chapters deal with small oscillations (with discussion of gyroscopic stabilisation), impulsive motion and the mechanics of continuous bodies, including the dynamics of strings and the propagation of discontinuities along them. In the opinion of the reviewer this is probably the most useful book available in this traditional field of mechanics, particularly for the reader interested in the mathematical structure of the subject. It is clearly written and easy to read, but the amount of detail covered is surprising in a single volume. Particular problems are worked out; there are no exercises for the student. In addition to a full table of contents, there is a short index—the reviewer would prefer it longer. There are some mis-

prints in addition to those noted on an errata slip, but none likely to cause confusion. *J. L. Synge* (Dublin).

Les limites de la loi de Newton. D'après un cours professé par Henri Poincaré. Bull. Astr. **17**, 121–178, 181–269 (1953).

Mise au point du cours professé par Poincaré à la Faculté des Sciences de Paris dans l'hiver 1906–1907, mise au point faite à partir des notes de Henri Vergne. (Extract from the Introduction by J. Chazy.)

Müller, Hans Robert. Zur Kinematik des Rollgleitens. Arch. Math. **4**, 239–246 (1953).

Grüss [Z. Angew. Math. Mech. **31**, 97–103 (1951); diese Rev. **13**, 80] hat den Sonderfall des Rollgleitens betrachtet, wobei eine Kurve K an einer raumfesten Kurve K' so abrollt und gleitet, dass die Bogenlängen der Kurven, gemessen zwischen den gemeinsamen Berührungspunkten zur Anfangszeit $t=0$ und einem Zeitpunkt t in einem von t unabhängigen Verhältnis λ stehen. Verfasser leitet die Formeln von Grüss in einfacher Weise her und gibt allgemeine Betrachtungen über den Bewegungsvorgang indem er die kennzeichende Differentialgleichung vollständig integriert. Aus der Anwendung auf ein Beispiel folgt der Satz: sollen die Flanken zylindrischer Zahnräder aufeinander rollgleiten, so müssen Sie als Trochoiden ausgebildet werden. Wir erwähnen noch die allgemeine Eigenschaft: λ ist das Doppelverhältnis der vier Punkte: Momentanpol, Berührungspunkt X von K und K', die Krümmungsmittelpunkte von K und K' in X. *O. Bottema* (Delft).

Signorini, A. Stereodynamische Anwendungen einer Erweiterung der Culmannschen Ellipse. Arch. Math. **4**, 154–162 (1953).

Ist m die Masse des starren Körpers K, S sein Schwerpunkt, ϵ eine beliebige Ebene, P die Orthogonalprojektion von S auf ϵ, E_s das Trägheitsellipsoid von K in P, s_ϵ die Schnittellipse von E_s und ϵ, σ_ϵ ihr Flächeninhalt, e_ϵ die Ellipse, die s_ϵ entspricht in der Homothetie mit Zentrum P und Faktor $\pi/\sigma_\epsilon m^{1/3}$, dann ist diese "Zentralellipse" e_ϵ eine Erweiterung der Cullmannsche Ellipse; sie fällt mit dieser zusammen wenn K eben ist und erlaubt bekannte Eigenschaften aus der ebenen Geometrie der Massen auf räumliche Systeme auszudehnen. Verfasser gibt davon mehrere Beispiele; wir erwähnen Bemerkungen über die instantane Bewegung des Körpers und die auftretenden Trägheitskräfte, sowie über die Staudeschen Achsen bei der Bewegung gewisser Systeme mit vier Freiheitsgraden.

O. Bottema (Delft).

Schallreuter, W. Das Prinzip des kleinsten Zwanges. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe **1**, 165–181 (1953).

A general discussion of the application of Gauss' principle of least constraint and the principle of Le Chatelier and Braun to a number of physical systems, mechanical, thermodynamic, and electrical. *J. L. Synge* (Dublin).

Aržanyh, I. S. Involution systems of zero rank. Akad. Nauk Uzbek SSR. Trudy Inst. Mat. Meh. **9**, 102–123 (1952). (Russian)

The Hamilton-Jacobi theorem is generalized to canonical systems where $H=H_0(q, p)+G(t, H_i)$, $i=1, \cdots, r$, the H_i being in involution and integrals of the system with the Hamiltonian H_0. The partial differential equation of the theorem is replaced by the set $H_j(q, \partial W/\partial q)=h_j$, $j=0, 1, \cdots, r$. The new theorem is specialized for generalized Liouville and Staeckel type systems, the generalization consisting here in the substitution of groups of variables for the single variables in terms of which the separation was originally defined for the mentioned types. It is impossible to say anything more about the paper without reproducing some involved formulas of prohibitive length.

A. W. Wundheiler (Chicago, Ill.).

Aržanyh, I. S. On a theorem of Hamilton-Jacobi type. Doklady Akad. Nauk SSSR (N.S.) **91**, 463–466 (1953). (Russian)

For the sake of brevity the notations and the wording of the theorem have been somewhat modified in this review. Consider $2n$ equations of the form $\dot{q}=\partial H/\partial p, \dot{p}=-\partial H/\partial q+\pi$ where $\pi=\Pi(q, p)$. Put $H(q, p)=H^*(q, \pi)$. Let V and W be functions of q (but not of p), and write V' and W' for their gradients (in q). If V satisfies the equation (1) $V=H^*(q, V')$ and the functions $\pi-V'$ are in involution (relative to the base q, p), then the equations $\Pi(q, W')=V'$ are compatible in W, and the equations of motion become $\partial(V-\dot{W})/\partial c=0$, where the c are the n constants of the complete integral of (1). This set contains n equations of the first order in q. Also $p=W'$. *A. W. Wundheiler* (Chicago, Ill.).

De Cicco, John. Conservative physical systems of curves upon a surface. Univ. Nac. Tucumán. Revista A. **9**, 23–36 (1952).

The author considers the situation in which a surface Σ is immersed in a three-dimensional space which is the seat of a conservative field of force. A system S_k of curves on Σ is a system of ∞^1 curves such that a constrained motion of a particle along any one of the curves is possible, with the pressure P and the normal component of force N being related by the equation $P=kN$. Here $k(\neq -1)$ is a constant. The paper is devoted largely to the derivation of the fundamental equations of the theory of systems S_k. Analogues or generalizations of Lagrange's equations and of the Hamiltonian canonical equations appear, as do also the equations defining certain two-parameter subfamilies of an S_k, called natural families. Some of the more immediate geometrical consequences of the equations are discussed.

L. A. MacColl (New York, N. Y.).

Vogel, Théodore. Systèmes dynamiques héréditaires à déferlement. Rend. Sem. Mat. Univ. Padova **22**, 64–80 (1953).

This paper is concerned with the following situation. A dynamical system is moving in accordance with a law expressed by a certain system of differential equations. When the state of the system attains a certain critical condition the law is abruptly changed to another. Subsequently there may be an abrupt return to motion under the first law; and this alternation of motions may continue indefinitely, or ultimately the system may move indefinitely in accordance with one of the laws. The author gives a brief, summary, and not entirely clear account of the results of studies of such motions. It is brought out that many motions of this kind can be found in engineering and physics. It is also shown that such motions can be regarded, in a certain sense, as limiting case of motions which were studied by Volterra by means of integro-differential equations. *L. A. MacColl* (New York, N. Y.).

Saxon, David S., and Cahn, A. S. Modes of vibration of a suspended chain. Quart. J. Mech. Appl. Math. **6**, 273–285 (1953).

The authors consider an inextensible chain suspended between two points at the same level and vibrating with small amplitude in the equilibrium plane. A method for computing approximations to the characteristic frequencies is presented. The numerical results of the computations are tabulated and graphically compared with the corresponding experimental data. *H. D. Block* (Minneapolis, Minn.).

Grammel, R. Die stationären Bewegungen des selbsterregten Kreisels und ihre Stabilität. Ing.-Arch. **21**, 149–156 (1953).

Considering a (not necessarily symmetrical) top subjected to a torque which is represented by a vector fixed in the body, the author studies motions such that the angular velocity is represented by a vector which is also fixed in the body. Motions of this kind are found, and their stability is investigated. Since there are numerous cases to be considered, no concise summary of the results is possible.

L. A. MacColl (New York, N. Y.).

Linnik, Yu. V., and Novoselov, V. S. Random disturbances of the regular precession of a gyroscope. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 361–368 (1953). (Russian)

Consider a system

$$(1)\quad \frac{dx_i}{dt}=\lambda_i[x_j, A_k(t), t]+S_i[x_j, A_k(t), t] \quad (i, j=1, \cdots, n)$$

where the functions $A_k(t)$ $(k=1, \cdots, l)$ characterize an l-dimensional random process. Let $a_k(t)$ denote the mathematical expectation of $A_k(t)$ and let $A_k(t)=a_k(t)+b_k(t)$, so that the mathematical expectation of the random functions $b_k(t)$ is zero. Further, let $S=\{S_i[x_j, A_k(t), t]\}$ denote an n-dimensional random vector-function which for the given values of the arguments characterizes an n-dimensional random process. The initial data of the system (1) are assumed to be random and given by a distribution with probability density

$$P\{\xi_i^0 \leqq x_i^0-y_i^0<\xi_i^0+d\xi_i^0\}=f(\xi_i^0, \cdots, \xi_n^0)d\xi_i^0\cdots d\xi_n^0,$$

where y_i^0 denotes the mathematical expectation of x_i^0. The solution of (1) is some n-dimensional random process $x(t)=\{x_i(t)\}$.

The first part of this paper investigates the distribution of the deviations of the solutions of (1) from $y_i(t)$, where $y_i(t)$ denotes that solution of (1) for which

$$x_i^0=y_i^0, \quad A_k(t)=a_k(t), \quad S\{S_i[x_j, A_k(t), t]\}=0$$

holds. Consider a finite time interval and assume that the probable values of $\max|b_k(t)/a_k(t)|$ and $\max|(x_i-y_i)/y_i|$ are small in this interval. The random process characterized by the functions $S_i[x_j, A_k(t), t]$ may be considered as a collection of random surfaces having the property that the probability is unity that these surfaces have bounded partial derivatives $\partial S_i/\partial x_j$ and $\partial S_i/\partial A_k$. Under these assumptions the system (1) can be linearized and reduced to the form

$$(2)\quad \frac{dz_i}{dt}=\sum_{j=1}^{n}X_{ij}(t)z_j+F_i(t) \quad (i=1, \cdots, n),$$

where

$$z_i=x_i-y_i, \quad F_i(t)=\varphi_i(t)+S_i(t), \quad S_i(t)=S_i[y_j(t), a_k(t), t].$$

The functions $\varphi_i(t)$ and $X_{ij}(t)$ can be easily evaluated and the system (2) solved with given initial values z_i^0.

Introduce one column matrices $B(t)$, $S(t)$ and z^0 with the elements $b_k(t)$, $S_i(t)$ and z_i^0 respectively, and assume that they represent statistically independent processes, $B(t)$ and $S(t)$ being, in addition, stationary. In order to secure continuity of the processes their correlation matrices are assumed to be continuous. Further, the processes $B(t)$, $S(t)$ and z^0 are assumed to have Gaussian distributions. Then $z=\{z_i(t)\}$ is also a Gaussian process with correlation matrix R, and probability density

$$(3)\quad f(\xi_1, \cdots, \xi_n)=[2^n\pi^n D(R)]^{-1/2} \times\exp\left[(-1/2D(R))\sum_{i,j=1}^{n}D_{ij}\xi_i\xi_j\right],$$

where $D(R)$ is the determinant of the matrix R and D_{ij} are the algebraic complements of its elements.

The second part of the paper is concerned with application of the results obtained to the motion of a gyroscope. Let θ, ψ, φ be the angles of nutation, precession and proper rotation of a gyroscope, respectively. Furthermore, let A and C be the moments of inertia of a gyroscope, m its mass and l the distance of its center of gravity from the fixed point. Then the Lagrangian equations of motion can be put in the form

$$\ddot{\theta}=\dot{\psi}^2\sin\theta\cos\theta-(C/A)(\dot{\varphi}+\dot{\psi}\cos\theta)\dot{\psi}\sin\theta+(mgl/A)\sin\theta+M_\theta/A,$$

$$\ddot{\psi}=[C(\dot{\varphi}+\dot{\psi}\cos\theta)\cos\theta-2A\dot{\psi}\dot{\theta}\cos\theta]/A\sin\theta+M_\psi/A\sin^2\theta,$$

$$d/dt(\dot{\varphi}+\dot{\psi}\cos\theta)=M_\varphi/C,$$

where the moments M_θ, M_ψ and M_φ are assumed to be random functions of the variables θ, $\dot{\theta}$, ψ, $\dot{\psi}$, t. For the given initial conditions θ_0, $\dot{\theta}_0=0$, ψ_0, $\dot{\psi}_0$, satisfying the condition,

$$(4)\quad (A-C)\dot{\psi}_0^2\cos\theta_0-C\dot{\varphi}_0\dot{\psi}_0+mlg=0,$$

and for $M_\theta=M_\psi=M_\varphi=0$, this system has the unique solution $\theta=\theta_0$, $\psi=\dot{\psi}_0$, $\dot{\varphi}=\dot{\varphi}_0$, known as the regular precession of a gyroscope. The case $\sin\theta_0=0$ has been excluded from consideration.

Assume that the characteristics of a gyroscope and the initial data of its motion are $A=\bar{A}+b_1$, $C=\bar{C}+b_2$, $ml=\overline{ml}+b_3$, and $\theta_0+z_1^0$, $\dot{\theta}_0+z_2^0$ $(\theta_0=0)$, $\psi_0+z_3^0$, $\dot{\varphi}_0+z_4^0$, respectively, where the bar denotes a mean value, and b_k and z_i^0 are independent random variables with given Gaussian distributions. Furthermore, assume that the mean values satisfy the condition of regular precession (4), and that $M_\varphi=0$. Put $\theta=x_1$, $\dot{\theta}=x_2$, $\psi=x_3$ and $z_4^0=b_4$, write down the linearized system corresponding to (2), and the correlation matrices of the processes $B(t)$, $S(t)$ and z_0. Then the correlation matrix R of the process $z=\{z_i(t)\}$ can be exhibited. Finally the density of the distribution of the deviations of the motions of a gyroscope from the regular precession can be evaluated by formula (3). *E. Leimanis.*

Hydrodynamics, Aerodynamics, Acoustics

Kirpičev, M. V. Dimensional analysis. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1953**, 1330–1340 (1953). (Russian)

Ericksen, Jerald Laverne. Characteristic surfaces of the equations of motion for non-Newtonian fluids. Z. Angew. Math. Physik **4**, 260–267 (1953).

L'auteur dérive la condition caractéristique pour les équations de Rivlin du mouvement d'un fluide généralisé

et il détermine les directions caractéristiques dans le cas du mouvement plan. *R. Gerber* (Toulon).

Sternberg, E., and Sadowsky, M. A. Axisymmetric flow of an ideal incompressible fluid about a solid torus. J. Appl. Mech. **20**, 393–400 (1953).

This paper contains an exact solution to the problem presented by irrotational axisymmetric flow past a torus of circular cross section. At infinity the motion is assumed uniform and parallel to the axis of the torus. Using toroidal coordinates, the velocity potential and stream function are both obtained in series form in terms of Legendre functions of fractional order (for which tables are available). Convergence of the series, although not rigorously established, is found to be unusually favorable for numerical work, and a complete streamline pattern for one shape-ratio is included. As the authors remark, exact solutions have previously been given by Hicks and by Dyson, but in forms unsuited to numerical evaluation. This paper is thus an important contribution to the problem, and moreover, as the individual component solutions are related to the source ring and vortex ring, it may have interest even outside the present application. *J. B. Serrin* (Cambridge, Mass.).

Nekrasov, A. I. Determination of the two-dimensional potential motion of an incompressible fluid by giving the values of the modulus of its velocity. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 483–484 (1953). (Russian)

Let the velocity be $qe^{i\theta}$. Since $-\theta$ and $\log q$ must be conjugate harmonic functions, a complex potential with given q is readily found by quadratures. *J. H. Giese.*

***Szebehely, V. G. A measure of unsteadiness of time dependent flows.** Proceedings of the Third Midwestern Conference on Fluid Mechanics, University of Minnesota, March 23, 24, and 25, 1953, pp. 221–231. University of Minnesota, Minneapolis, Minn., 1953. $6.00.

See the following report: The David W. Taylor Model Basin, Washington, D. C., Rep. **833** (1952); these Rev. **14**, 810.

Longuet-Higgins, M. S. On the decrease of velocity with depth in an irrotational water wave. Proc. Cambridge Philos. Soc. **49**, 552–560 (1953).

The author proves the following for irrotational surface waves of finite amplitude in a uniform incompressible fluid. In any space-periodic motion the mean square velocity (or any velocity-component) decreases and the mean dynamic pressure increases with depth. If the depth is infinite the velocity tends exponentially to zero with depth (in a suitable coordinate system). In a progressive wave in water of uniform depth the mass-transport velocity decreases with mean depth of a particle. Similar results for the solitary wave. The results on mass transport were also given by F. Ursell [same Proc. **49**, 145–150 (1953); these Rev. **14**, 594], but with a more complicated proof.

J. V. Wehausen (Providence, R. I.).

Long, Robert R. Some aspects of the flow of stratified fluids. I. A theoretical investigation. Tellus **5**, 42–58 (1953).

L'auteur étudie certaines propriétés du mouvement d'un fluide incompressible, pesant, non homogène. Le mouvement considéré est un écoulement dans un plan vertical, limité par deux parois rigides, et il est supposé exister à l'infini amont un gradient de vitesse et un gradient de densité. Les équations exactes du mouvement sont formées; elles se ramènent à une équation du second ordre, en général non linéaire, et les solutions sont étudiées du point de vue de l'unicité et de la stabilité. Une condition suffisante de stabilité est établie et il apparait sur des cas particuliers que cette condition entraîne l'instabilité de l'écoulement. Il semble que cette propriété, que l'unicité et l'instabilité d'une solution se correspondent, soit générale pour les mouvements en cause. Cette condition de plus exprime qu'une certaine quantité énergétique liée à l'écoulement est extrêmum. Le cas où la vitesse est uniforme en amont est étudiée en détail. Il y a alors deux régimes du mouvement suivant la valeur d'un nombre de Froude, et de nombreuses analogies avec les écoulements à surface libre dans un canal sont mises en évidence. *R. Gerber* (Toulon).

Goodman, Theodore R. The upwash correction for an oscillating wing in a wind tunnel. J. Aeronaut. Sci. **20**, 383–386, 406 (1953).

The title problem is solved for incompressible flow in a circular tunnel, the analysis being condensed from a more extensive treatment, including two-dimensional tunnels [Cornell Aeronaut. Lab. Rep. **AD—744—W-1** (1951)]. The results are derived from an acceleration potential, using Fourier integral techniques. A chart is presented giving the complex downwash correction as a function of reduced frequency at various streamwise positions up to 1.5 tunnel radii downstream of the wing leading edge. (The reduced frequency parameter R in this chart is not defined in the paper but must be presumed equal to k. Also, the decimal point is misplaced for values of the abscissa greater than unity.) *J. W. Miles* (Los Angeles, Calif.).

Gerber, Robert. Observations sur un travail récent de M. Miche. J. Math. Pures Appl. (9) **32**, 79–84 (1953).

R. Miche [same J. (9) **28**, 151–179 (1949); these Rev. **11**, 62] has claimed to derive Navier's dynamical equation for viscous fluids from a variational principle. The author shows that various steps in the derivation are faulty and that all that can be salvaged is a certain identity satisfied by the vector potential of a velocity field satisfying the compatibility condition for Navier's equation. The author shows also that a likely substitute for Miche's variational proposal does not yield the desired result. *C. Truesdell.*

***Bourrières, François-Joseph. Les autoplissements et les équations intrinsèques des fluides visqueux.** Publ. Sci. Tech. Ministère de l'Air, no. 279, Paris, 1953. i+95 pp. 1200 francs.

The author asserts that experiments indicate that in a jet of fluid surrounded by fluid of the same type, the streamlines are lines of constant speed. The parabolic velocity distribution appropriate to the pipe whence the jet issues must then continue along the jet, assumed only slightly curved. To reconcile such a phenomenon with Navier's equation, the author observes that volume forces are required. By a method of approximation which the reviewer cannot follow, the author replaces these by the action of the surrounding fluid, thus obtaining an analogy to the buckling of a column and calculating the resulting sinusoidal form of some types of jet. There is a good deal of experimental work reported in the paper. [In detailing a calculation of the paths of the particles in a potential flow past a cylinder, the author is apparently unaware of the work of Maxwell, Scientific papers, v. 2, Cambridge, 1890, pp. 208–214.] *C. Truesdell* (Bloomington, Ind.).

Tomotika, S., Aoi, T., and Yosinobu, H. On the forces acting on a circular cylinder set obliquely in a uniform stream at low values of Reynolds number. Proc. Roy. Soc. London. Ser. A. **219**, 233–244 (1953).

By the Oseen approximation it is shown that the velocity components in the plane normal to the axis of a yawed infinite circular cylinder can be determined independently of the component parallel to the axis of the cylinder. These velocity components can then be deduced from the results of the unyawed case and the third component can be obtained by a simple integration. The total force on the cylinder has been calculated by resolving it into transverse and longitudinal components. It is found that the force due to tangential shear stresses are generally not in the direction of the free stream but tends in that direction as the Reynolds number approaches zero. *Y. H. Kuo* (Pasadena, Calif.).

Tomotika, S., and Aoi, T. The steady flow of a viscous fluid past an elliptic cylinder and a flat plate at small Reynolds numbers. Quart. J. Mech. Appl. Math. **6**, 290–312 (1953).

Noticing errors in several previous papers, the authors recalculate the viscous flow past an elliptic cylinder by Oseen's approximation. The drag is also correct to the third order in Reynolds number. As two limiting cases, flows around a flat plate placed parallel to, and perpendicular to the stream are considered in detail. It is interesting that the flow pattern around a flat plate placed edgewise along the free stream is symmetrical at low Reynolds numbers.

Y. H. Kuo (Pasadena, Calif.).

Sidrak, Sobhy. The drag on an elliptic cylinder, of small eccentricity, in a stream of viscous liquid, at small Reynolds numbers. Proc. Math. Phys. Soc. Egypt **4** (1952), no. 4, 17–27 (1953).

The author reconsiders the problem of drag on an elliptic cylinder in a viscous liquid by the Oseen approximation. The evaluation of the constants appearing in the expression for drag is carried to the third order in Reynolds number. This paper should be studied in conjunction with the author's earlier paper [Proc. Roy. Irish Acad. Sect. A. **53**, 65–81 (1950); these Rev. **12**, 137] and the paper by Tomotika and Aoi reviewed above. *Y. H. Kuo.*

Kočina, N. N. Some problems on the spatial spreading of ground water. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 377–381 (1953). (Russian)

Filtration in an infinite homogeneous medium is considered. The free surface (water table) of ground water is assumed to have an initial form $z=F(x, y)$. The author first writes the expression for free surface at time t in integral form, and then gives three examples: (a) $F(x, y)=\delta$ when $|x|<a$, $|y|<b$, zero otherwise; (b) $F(x, y)=\delta$ when $r<a$, zero otherwise; (c) $F(x, y)=\delta \cos \alpha x \cos \beta y$. In a fourth example she introduces the effect of evaporation. No derivations are given, but short discussions accompany the first two examples. *R. E. Gaskell* (Seattle, Wash.).

Davies, T. V. The forced flow of a rotating viscous liquid which is heated from below. Philos. Trans. Roy. Soc. London. Ser. A. **246**, 81–112 (1953).

The author explores the hydrodynamical theory of the Fulz experiment in which a dishpan 15 cm. in radius is filled with water to a depth of 2 cm., heated symmetrically round the outer edge of its base and rotated with constant angular speed about its axis of symmetry, which is vertical. In his theory, the author takes account of the viscosity of the fluid and assumes that the flow of heat within the fluid is by molecular conduction. When there is no heating, the motion of the fluid is one of solid rotation, the heating producing departures from this state of motion that can be treated as perturbations. The equations controlling the perturbed motion are the three equations of motion, the equation of continuity, the equation of heat transfer and an equation stating that the perturbation density and temperature are proportional to one another. The author then solves the following sequence of problems: (a) no rotation, steady motion, no non-linear terms in the fluid acceleration, no zonal variations in the three components of velocity and no net heating of the fluid; (b) same conditions as in (a) except that rotation and net heating are included; (c) consideration of the effects of each of the non-linear terms $w\partial v/\partial z$, $ur^{-1}\partial(vr)/\partial r$ and v^2r^{-1}, where u, v, w are the radial, zonal and vertical velocities, respectively, on the motion in case (b). Methods of approximation are employed which are shown to be amply adequate for the conditions of the experiment.

The march of u, v and w with position of the fluid is described in detail and it turns out that the relative flow tends to become zonal in character as the speed of rotation increases. The relative zonal motion is principally in the direction of rotation, but when the non-linear terms are allowed for, the relative zonal motion in the outer parts of the fluid is in the sense opposed to the rotation. The analogy with the westerly air-flow on the earth, corresponding to motion in the sense of the earth's rotation, and the easterly flow in the opposite sense, is indicated. *G. C. McVittie.*

Chandrasekhar, S. The instability of a layer of fluid heated below and subject to Coriolis forces. Proc. Roy. Soc. London. Ser. A. **217**, 306–327 (1953).

The aim of this paper has been to examine the stability of a horizontal layer of fluid heated below, subject to a gravitational acceleration g normal to the layer, and to the Coriolis force resulting from a rotation with an angular velocity Ω about a direction making an angle ϑ with the vertical. It is shown that the effect of the Coriolis force is to inhibit the onset of convection, the extent of the inhibition depending on the value of a non-dimensional parameter $T=4d^4\Omega^2\nu^{-2}\cos^2\vartheta$, where d denotes the depth of the layer and ν is the kinematic viscosity. Tables of the critical Rayleigh numbers R_c for the onset of the convection are provided for the case of (1) both bounding surfaces free; (2) both bounding surfaces rigid; and (3) one surface free, the other rigid. In all three cases $R_c \sim T^{2/3}$ as $T \to \infty$; the critical temperature gradient $-\beta_c$ for the onset of convection being given by the equation $g\alpha\beta_c \sim \kappa(\Omega^4 \cos^4\vartheta/d^4\nu)^{1/3}$, where κ denotes the coefficient of thermometric conductivity and α the coefficient of volume expansion. The question whether thermal instability can set in as oscillations of increasing amplitude is examined for case (1), and it is shown that if $\kappa/\nu<1.478$ this possibility does not arise; but if $\kappa/\nu>1.478$, overstability is the first type of instability to arise for all T's greater than a certain determinate value. *Z. Kopal.*

***Chandrasekhar, S. Some aspects of the statistical theory of turbulence.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 1–17. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

Dans les deux premiers paragraphes, l'auteur rappelle les résultats fondamentaux de Robertson et les siens propres

sur l'application de la théorie des invariants à la turbulence isotrope ou axisymétrique. Il applique ensuite les méthodes générales, dans le cas de la turbulence isotrope, aux problèmes suivants: 1) phénomènes magnétiques dans un fluide turbulent conducteur; 2) fluctuations de densité dans un fluide turbulent compressible (les corrélations de densité, ramenées à la densité moyenne, se propagent avec la vitesse $(2c^2+\frac{2}{3}\overline{u^2})^{1/2}$, où c est la vitesse du son et $\overline{u^2}$ la valeur quadratique moyenne de la vitesse turbulente); 3) cas d'un fluide dans lequel les effets de gravitation dûs aux fluctuations de densité sont importants. *J. Bass* (Chaville).

***Lin, C. C. A critical discussion of similarity concepts in isotropic turbulence.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 19–27. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

Rappel de résultats sur les lois de similitude du spectre de la turbulence derrière une grille. Domaine de validité de ces lois. Forme correspondante de la loi de décroissance de l'intensité de la turbulence. Discussion de la stabilité du spectre, envisagée du point de vue suivant: la turbulence étant supposée décroître de telle sorte que le spectre satisfasse à la loi de similitude, et évolue conformément à l'équation de transfert de Heisenberg, comment réagit il à l'effet d'une perturbation, limitée à une étroite bande de fréquences, qui en modifie la forme? *J. Bass* (Chaville).

Lin, C. C. Note on the mean square value of integrals in the statistical theory of turbulence. Quart. Appl. Math. 11, 367–370 (1953).

Calcul du carré moyen de l'intégrale $\int p dS$ sur une sphère, p étant la pression d'un fluide turbulent isotrope. Cas des petites sphères et des grandes sphères. Remarques sur l'extension de la formule trouvée à des intégrales étendues à des domaines plus compliqués ou à plus de deux dimensions. *J. Bass* (Chaville).

Eckart, Carl. Relation between time averages and ensemble averages in the statistical dynamics of continuous media. Physical Rev. (2) 91, 784–790 (1953).

Les corrélations (spatio temporelles) d'une fonction $p(x, t)$ sont définies par

$$\langle pp' | \tau \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} p(x, t) p(x', t-\tau) dt.$$

L'auteur suppose que p vérifie une équation fonctionnelle linéaire et homogène $\partial p/\partial t = Lp$ ou linéaire et non homogène $\partial p/\partial t + a(x, t) = Lp$, L opérateur linéaire donné, et il étudie quelques propriétés des corrélations de leurs solutions. Application au problème (d'ailleurs régi par une équation d'un type différent) de la dispersion de la lumière dans un milieu d'indice variable. Quelques références bibliographiques (qui pourraient être plus complètes; voir en particulier: G. K. Batchelor, The theory of homogeneous turbulence, Cambridge Univ. Press, 1953, pp. 55–75, 92–98; ces Rev. 14, 597; H. W. Liepmann, Z. Angew. Math. Physik 3, 321–342, 407–426 (1952), pp. 330–340, 420; ces Rev. 14, 597). *J. Bass* (Chaville).

Tatarskiĭ, V. I. On the fluctuations of phase of sound in a turbulent medium. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1953, 252–258 (1953). (Russian)

This paper considers the question of fluctuations of phase of sound received by two microphones situated in a turbulent medium. The mean quadratic fluctuations of the difference of phases is computed for an arbitrary relation between a quantitative base and the distance traversed by the sound in the turbulent medium. Treatment of the question is carried out within the framework of geometrical acoustics. *Author's summary.*

Uberoi, Mahinder S. Eddy turbulence and random sound in a compressible fluid. Proc. Cambridge Philos. Soc. 49, 731–734 (1953).

Un tenseur du second ordre $Q_{ij}(\xi; \mathbf{x})$ (tenseur de corrélation d'un champ de vitesse aléatoire en deux points ξ et $\xi+\mathbf{x}$) peut être décomposé en une somme

$$Q_{ij}(\xi; \mathbf{x}) = A_{ij}(\xi; \mathbf{x}) + B_{ij}(\xi; \mathbf{x});$$

A_{ij} a une divergence nulle et correspond à la turbulence proprement dite, B_{ij} a un rotationnel nul et correspond aux ondes acoustiques ["random noise" de J. Moyal, mêmes Proc. 48, 329–344 (1952); ces Rev. 14, 106]. Calcul de A_{ij} et B_{ij} en fonction de Q_{ij}. Cas de turbulence homogène et isotrope. *J. Bass* (Chaville).

McVittie, G. C. A method of solution of the equations of classical gas-dynamics using Einstein's equations. Quart. Appl. Math. 11, 327–336 (1953).

The author assumes that the metric tensor of space-time is orthogonal and that the four functions involved differ from the Minkowski metric by small terms proportional to the Einstein gravitational constant. The Einstein field equations then lead to explicit expressions for the density, pressure and velocity of a perfect gas in terms of these four functions. However, these four functions are not independent but must satisfy consistency relations which are derived in the paper. When the Newtonian approximation of the field equations is made, one obtains general solutions of the equations of classical gas dynamics for motions which are either of constant or of variable entropy in which the density times the velocity is the gradient of a scalar. As an illustration the method is applied to one-dimensional motions. *A. H. Taub* (Urbana, Ill.).

Sedov, L. I. On integration of the equations of one-dimensional motion of a gas. Doklady Akad. Nauk SSSR (N.S.) 90, 735 (1953). (Russian)

This paper exhibits a four-parameter family of inviscid time-dependent one-dimensional, cylindrical, or spherical flows involving an arbitrary function and a quadrature. *J. H. Giese* (Havre de Grace, Md.).

***Ferrari, Carlo. L'écoulement non permanent de fluides compressibles à travers des conduites droites. On the non-permanent flow of compressible fluids in straight ducts.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 273–296, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and English)

The problem considered is that of one-dimensional compressible flow in a tube, modified, however, to consider a small amount of lateral expansion of the (assumed elastic) tube under internal pressure. The treatment follows the usual pattern, making use of the Riemann invariants, with modifications to account for the elasticity of the tube. For a special value of the adiabatic constant the characteristics become straight lines. The discontinuities at a shock front are derived in the same general form as the usual one-dimensional conditions. The theory is applied in detail to the problem of fuel injection in a Diesel engine. Finally,

the author writes down the hydrodynamic equations under quite general conditions. The assumption of isentropy is avoided, energy is introduced from exterior sources, and a chemical reaction may be going on, as in an explosion or in burning. However, no application of this theory is made.

E. Pinney (Berkeley, Calif.).

Schäfer, Manfred. Eine einheitliche Charakteristikenmethode zur Behandlung gemischter Unterschall-Überschallströmungen. J. Rational Mech. Anal. **2**, 383–412 (1953).

Il s'agit d'un article consacré à l'étude de l'écoulement irrotationnel d'un fluide non visqueux. L'auteur nous y expose la théorie des caractéristiques; au lieu d'utiliser une représentation du plan de l'écoulement sur le plan de l'hodographe, on utilise ici les variables caractéristiques et la métrique correspondante dans le plan de l'écoulement; on peut ainsi éviter le léger inconvénient d'avoir plusieurs points du plan de l'écoulement susceptibles d'avoir une même image dans le plan de l'hodographe. Comme on le sait, les caractéristiques sont réelles si l'écoulement est supersonique et imaginaires conjuguées si l'écoulement est subsonique; dans ce dernier cas l'auteur remplace les variables caractéristiques par leur partie réelle et leur partie imaginaire. Selon l'auteur de cette analyse, cet article nous offre des méthodes "parallèles" pour traiter les problèmes supersoniques et subsoniques plutot qu'une méthode "unitaire" permettant de simplifier notablement le traitement des problèmes transsoniques. *P. Germain.*

***Martin, M. H., and Thickstun, W. R. An example of transonic flow for the Tricomi gas.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 61–73. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

L'exemple en question est celui de l'écoulement dans une tuyère de Laval, au voisinage du col, lorsque le fluide passe d'un régime subsonique à un régime supersonique. L'approximation utilisée est celle de Tricomi; il s'agit de trouver une solution de l'équation de Tricomi prenant une valeur à l'extérieur de la région limitée par deux caractéristiques concourantes en un point de la ligne sonique, et trois valeurs à l'intérieur de cette même région. La solution étudiée par les auteurs n'est pas nouvelle, mais l'étude des propriétés du gaz fictif, conduisant à l'équation de Tricomi, et de la structure de l'écoulement obtenu dans le plan physique est faite avec grand soin. En particulier, il est montré que la ligne sonique présente deux points d'enroulement, analogues aux points singuliers d'une spirale de Cornu.

P. Germain (Providence, R. I.).

***Busemann, Adolf. The nonexistence of transonic potential flow.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 29–39. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

La lecture d'un article du célèbre aérodynamicien A. Busemann présente toujours une assez grande difficulté, car l'auteur emploie des raisonnements assez spéciaux permettant de démontrer un théorème de façon pleinement satisfaisante à ses yeux; souvent la validité de la preuve, d'un point de vue strictement mathématique reste discutable, et en tout cas le lecteur moyen déconcerté a de la peine à suivre. Oserai-je avancer qu'un des modes de raisonnement de cet auteur est "l'analogie"; celle-ci alliée à une intuition remarquable et un jugement très sûr lui permet de faire des "sauts", tels que les scientifiques moins bien doués et plus conventionnels ne peuvent suivre. Mais, l'analogie est souvent très riche.

Cet article traite de l'impossibilité d'un écoulement potentiel continu dans le régime transsonique. En fait, cela signifie qu'un tel écoulement potentiel est excessivement rare et l'intérêt des études consacrées à ces écoulements se trouve ainsi diminué. L'objectif de l'auteur semble être de montrer le type de problèmes aux limites que l'on peut se poser dans un domaine mixte et de comparer ces problèmes aux problèmes classiques que l'on envisage dans le demi plan elliptique ou dans le demi plan hyperbolique. Toute l'analyse repose sur l'étude du comportement des solutions à variables séparées de l'équation $\psi_{ww}+t(w)\psi_{ww}=0$ dans une bande (w_1, w_2) et sur la description qualitative des variations de la solution lorsqu'on déforme le contour de cette bande dans la région hyperbolique. Ce que l'auteur veut dégager c'est la nécessité d'une "soupape" dans le contour situé dans la région hyperbolique et l'impossibilité de poser correctement un problème dans un domaine mixte en donnant les valeurs de la fonction inconnue tout le long d'un contour fermé.

P. Germain (Providence, R. I.).

Goroščenko, L. B. On calculation of the motion of a gas in a local shockfree supersonic zone. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 423–430 (1953). (Russian)

To find the irrotational plane flow around a circular cylinder the author sets the velocity potential

$$\varphi(r,\theta)=\sum_{n=0}^{\infty} r^{1-2n}\lambda_{2n}(\theta),$$

where λ_{2n} are defined as solutions of an infinite set of second order ordinary differential equations. He expands $\lambda_{2p}=\sum_{n=0}^{\infty}a_{2p,2n+1}\cos(2n+1)\theta$, takes $\lambda_0=V_\infty\cos\theta$, and calculates a few of the coefficients of λ_2 and λ_4, with which to compute the approximate velocity distribution. At Mach number $M_\infty=0.4065$ the maximum speed is sonic. Above $M_0=0.4973$ the approximation to V_{max}/V_0 becomes complex, interpreted to mean that above M_0 a shock appears. In support of this view the author verifies that near M_0 the criterion for the appearance of a shock developed by Nikolsky and Taganov [same journal **10**, 481–502 (1946) = NACA Tech. Memo. no. **1213** (1949); these Rev. **8**, 237; **10**, 639] is satisfied. *J. H. Giese.*

Cabannes, Henri. Méthode de détermination des ondes de choc détachées dans les écoulements stationnaires. C. R. Acad. Sci. Paris **236**, 1854–1856 (1953).

The author considers a body of revolution, of arbitrary shape, started impulsively from rest in a compressible fluid and moving in the direction of its axis of symmetry. The initial velocity and acceleration of the shock wave which is generated in front of the obstacle are found at all points along it, in terms of the initial velocity and acceleration of the obstacle, and the shape of the body. Studying in particular the motion of the point where the shock intersects the axis, the author evaluates, on the assumption that the acceleration of both shock and body remain constant, their relative position at the instant when the velocities of both become equal. He compares this with the experimentally observed distance between shock and body in steady flow [Heberle, Wood, and Gooderum, NACA Tech. Note no. **2000** (1950)]. The observed distance is greater by a factor 3.9 at a Mach number of 1.37, and by a factor 3.0 at a Mach number of 1.81. The author suggests that the ratio might be much closer to 1 at really high Mach numbers.

The author's formulae hold also for two-dimensional flow symmetrical about an axis if one of the terms is omitted.
M. J. Lighthill (Manchester).

Fraenkel, L. E. On the operational form of the linearized equation of supersonic flow. J. Aeronaut. Sci. **20**, 647–648 (1953).

Goldstein, S., and Ward, G. N. The linearised theory of conical fields in supersonic flow, with applications to plane aerofoils. Aeronaut. Quart. **2**, 39–84 (1950).

The concept of a conical field of supersonic flow is due to A. Busemann. Among those who applied this concept at an early stage were W. D. Hayes, H. J. Stewart, and one of the present authors, G. N. Ward. The present paper is perhaps the best exposition of this subject. It is distinguished by the parallel development of the analysis for the hyperbolic and elliptic regions of the field. The various configurations of a Delta wing relative to its apex Mach cone are considered one by one. In particular, the paper contains the analytic solutions of the problem for a yawed Delta wing, one of whose original leading edges has become a subsonic trailing edge. *A. Robinson* (Toronto, Ont.).

Chen, Yu Why. Flows through nozzles and related problems of cylindrical and spherical waves. Comm. Pure Appl. Math. **6**, 179–229 (1953).

The paper exhibits the solution to a problem already noted as outstanding in 1948 [R. E. Meyer, Quart. J. Mech. Appl. Math. **1**, 196–219, 451–469 (1948); these Rev. **10**, 338, 492]. The difficulty is to find what happens in an axisymmetrical supersonic flow through a nozzle which is of uniform diameter up to a certain point and then turns outwards through an angle, sending a fan of expansion waves in towards the axis. On the leading expansion wave the velocity gradient is discontinuous. This discontinuity grows as the axis is approached and becomes infinite like the inverse square root of distance from axis. The problem is to find what happens on the reflected Mach line. A purely linearized theory gives that the velocity gradient is logarithmically infinite. It is easy to check however that according to the exact equations of motion such an infinity on a characteristic cannot in fact be present.

The author treats the problem by the use of "characteristic variables" σ and τ which are constant along characteristics. He finds that the derivatives of the velocities with respect to τ become logarithmically infinite on the reflected characteristic $\tau=0$, $\sigma>0$. However the derivatives of the space-coordinates x and y with respect to τ have also such a logarithmic infinity, in such a way that at each point on the reflected characteristic finite velocity gradients with respect to x and y exist and are continuous. However these velocity gradients tend to infinity more rapidly (like the inverse first power of distance from the axis) than does the discontinuity in velocity gradient on the upstream Mach line. They are expansive waves in the case considered above.

The analysis is too complicated to be satisfactorily abstracted. It is set out for very general wave focussing problems as well as particularised to the case described above.
M. J. Lighthill (Manchester).

Van Dyke, Milton D. On supersonic flow past an oscillating wedge. Quart. Appl. Math. **11**, 360–363 (1953).

The author generalizes the work of Carrier [J. Aeronaut. Sci. **16**, 150–152 (1949); these Rev. **10**, 493] on the oscillations of a wedge in a supersonic stream to permit oscillations about other pivots than the apex. For slow oscillations he shows how the boundary of neutral stabilizing effect of the aerodynamic damping moment about the pivot, for a wedge aerofoil of 5° semi-vertex angle, depends upon the free-stream Mach number and the position of the pivot. He also shows, for comparison, the corresponding results from linear and second-order theory. *D. C. Pack* (Glasgow).

Miles, John W. A note on subsonic edges in unsteady supersonic flow. Quart. Appl. Math. **11**, 363–367 (1953).

In an earlier paper [same Quart. **11**, 1–8 (1953); these Rev. **14**, 815] the author gave the solution of the problem of supersonic flow over a rectangular wing tip and he extended the discussion to the case of an oblique subsonic leading edge adjacent to a supersonic leading edge. Here he makes a further extension of this work and finds the pressure distribution for a wing with a supersonic leading edge and a subsonic trailing edge. *D. C. Pack* (Glasgow).

Lawrence, H. R. The aerodynamic characteristics of low aspect ratio wing-body combinations in steady subsonic flow. J. Aeronaut. Sci. **20**, 541–548 (1953).

The title problem is attacked by an extension of the author's treatment of low-aspect-ratio subsonic wings [same J. **18**, 683–695 (1951); **20**, 218–219 (1953)], thereby filling a gap between Multhopp's application [Luftfahrtforschung **18**, 52–66 (1941) = NACA Tech. Memo. no. **1036** (1942)] of Prandtl's lifting line theory to the high-aspect-ratio problem and Ward's result for the limiting case of very low aspect ratio [Quart. J. Mech. Appl. Math. **2**, 75–97 (1949); these Rev. **10**, 644]. The analysis does not require the wing to be at the same inclination as the body but is limited to cylindrical bodies of circular cross-section, although the removal of this restriction fore and aft (but not along the root section) of the wing is discussed. The wing may have arbitrary twist but is restricted to have a straight trailing edge. *J. W. Miles* (Los Angeles, Calif.).

Li, Ting-Yi, and Nagamatsu, Henry T. Similar solutions of compressible boundary-layer equations. J. Aeronaut. Sci. **20**, 653–655 (1953).

Under the assumptions of viscosity coefficient directly proportional to the absolute temperature, Prandtl number of unity, constant specific heats, and constant surface temperature, the authors extend some results (Case (3) below) of Stewartson [Proc. Roy. Soc. London. Ser. A. **200**, 84–100 (1949); these Rev. **11**, 553] and show that similar solutions of the compressible boundary-layer equations are possible in the following cases:

(1) $$V_1=[(1+2q)(K_3z+C)]^{1/(1+2q)},$$

(2) $$V_1=ce^{K_1z},$$ (3) $$V_1=Cz^{K_3},$$

where q, K_3, C, c, K_1 are arbitrary constants, and V_1 and z are defined by

$$V_1=\frac{a_0}{a_1}u_1, \quad z=\int_{x_0}^{x}\left(\frac{a_1}{a_0}\right)^{(3\gamma-1)/(\gamma-1)}dx.$$

The subscripts 0 and 1 correspond to reference and local free-stream conditions, x is the distance along the surface, a is the velocity of sound, and u_1 the local free-stream velocity. Numerical results are also given for various values of the parameter K_3 and of the ratio of surface to stagnation temperature in case (3). *D. W. Dunn.*

Mawardi, Osman K. On a variational principle in acoustics. Acustica 3, 187–191 (1953).

This paper deals with the propagation of sound inside an acoustic labyrinth, consisting of a tube of rectangular cross-section interrupted at periodic intervals by thin partitions with staggered apertures. A Green's function is constructed. The solution makes use of the concept of the propagation constant, assuming that the higher modes are rapidly attenuated. Using Green's function, an integral representation of the solution is obtained. The propagation constant is then expressed by the ratio of two definite positive quantities and thus an expression involving the propagation constant is found, which is stationary for first order variations of the solution. By means of suitable trial functions a rapid evaluation of these stationary values is obtained. The results are shown numerically in two graphs.

M. J. O. Strutt (Zurich).

Četaev, D. N. On the acoustic resistance of a moving plane emitter. Doklady Akad. Nauk SSSR (N.S.) 90, 355–358 (1953). (Russian)

Extending his work on the case of a stationary medium [Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 439–444 (1951); these Rev. 13, 402], the author now makes the piston oscillate into a uniformly flowing medium, for which case the potential has been expressed by S. S. Voĭt [ibid. 17, 157–164 (1953); these Rev. 14, 1142]. The acoustic resistance is studied first generally, and then for a rectangular piston. The rather complicated results are used to construct numerical tables for the square case. *F. V. Atkinson.*

Anders, Till. Beugung akustischer Wellen an einer kleinen kreisförmigen Öffnung. Z. Physik 135, 219–224 (1953).

Similar to a method employed by Hönl (Fourier integral method) [Z. Physik 131, 290–304 (1952); these Rev. 13, 801] the problem of the diffraction of plane acoustic waves for perpendicular incidence on a circular aperture (radius R) in an infinite rigid plane is reduced to the problem of the solution of a dual integral equation

$$\text{(a)} \qquad \int_0^\infty \rho h(\rho) J_0(kr\rho)\,d\rho = \frac{1}{2\pi}, \quad 0<r<R,$$

$$\text{(b)} \qquad \int_0^\infty \rho\delta(\rho) h(\rho) J_0(kr\rho)\,d\rho = 0, \quad r>R,$$

where $\delta(\rho)=(1-\rho^2)^{1/2}$ for $\rho\leqq 1$ and $i(\rho^2-1)^{1/2}$ for $\rho>1$. With the solution $h(\rho)$ of (a), (b) the velocity potential of the diffracted and reflected field respectively is given by

$$u^+ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\beta) e^{ik(\alpha x+\beta y+\gamma z)}\,d\alpha d\beta,$$

$$u^- = e^{-ikz} - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\beta) e^{ik(\alpha x+\beta y-\gamma z)}\,d\alpha d\beta,$$

$$\alpha=\sin\theta\cos\phi, \quad \beta=\sin\theta\sin\phi, \quad \gamma=\cos\theta, \quad \rho=\sin\theta,$$

$$d\alpha d\beta = \rho\, d\rho\, d\phi.$$

A solution of (a), (b) is written as

$$\text{(c)} \qquad \rho\delta(\rho)h(\rho) = \sum_{n=0}^{\infty} C_n \rho^{1/2} J_{2n+\frac{1}{2}}(kR\rho)$$

and the C_n are defined as the solutions of an infinite system of linear equations. For a small aperture ($kR\ll 1$) an approximative value for the first coefficient C_0 is given.

F. Oberhettinger (Washington, D. C.).

Elasticity, Plasticity

Truesdell, C. Corrections and additions to "The mechanical foundations of elasticity and fluid dynamics". J. Rational Mech. Anal. 2, 593–616 (1953).

As the title indicates, this is a list of corrections and additions to an earlier paper [same J. 1, 125–171, 173–300 (1952); these Rev. 13, 794], which is a critical and scholarly review of the literature dealing with nonlinear theories of elasticity and fluid dynamics.

Only very minor modifications have been made in the introductory chapter. Chapter II, which covered a relatively unexplored topic in invariant theory, isotropic functions of tensors, has been expanded slightly by the addition of a more complete treatment of some of the scalar invariants of second order tensors which are used in continuum theories. Part of the third chapter was devoted to a careful analysis of the kinematical quantities which occur in the theories discussed. The author has added a modern derivation of Cauchy's local decomposition of a finite deformation into a translation, a rigid rotation, and a pure homogeneous deformation. A clearer treatment of strain and reciprocal strain ellipsoids is given. Also, the author includes an interesting discussion of what he calls the absolute time flux of a vector, the vanishing of which is a necessary and sufficient condition that the vector be invariant under the motion considered, and indicates how one can define an analogous operation for tensors of arbitrary order. The remainder of this chapter was devoted to a discussion of the mechanical and thermodynamical principles underlying most continuum theories. Though some references have been added, this material is essentially unaltered.

To Chapter IV, which contained an unusually comprehensive survey of the theoretical work which has been done in nonlinear elasticity, has been added a discussion of Green's and Shield's work on small twist superimposed on large biaxial extension and Rivlin's method of solving problems within the framework of the second-order theory. Among the references added are those to Birch's and Murnaghan's work on anisotropic materials, and to the work of Novozhilov, who has discussed the approximations made in deriving theories of plates, shells, etc., from the general theory. The author's rate-of-deformation theory of elasticity, according to which the material derivative of the stress tensor is a function of the stress and rate-of-deformation tensors, has been modified by replacing the material derivative by the absolute time flux. The theory now predicts that, when the material moves as a rigid body, the stress components relative to axes fixed in the material do not change, as it should. The problems of hydrostatic pressure and simple extension are discussed on the basis of the modified theory, but the latter solution is in error, and should be reworked using weaker assumptions concerning the time dependence of the motion. References to relevant experimental data have been added to Chapter V, in which the author exploited the methods of tensor and dimensional analysis to reduce theories of fluid dynamics based on very general assumptions to a definite form. *J. L. Ericksen.*

***Hershey, A. V. A review of the definitions of finite strain.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 473–478. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author claims advantages in using a strain tensor defined as that symmetric tensor whose principal axes

coincide with spatial principal axes of strain and whose proper values are the principal extensions. This idea is not new, and the author does not mention the serious objection that the off-diagonal components become, in general, infinite series in the displacement gradients. There is a discussion of other definitions of strain. [Cf. §§14–17 of the reviewer's paper in J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); 2, 593–616 (1953); these Rev. 13, 794, and the preceding review.] *C. Truesdell* (Bloomington, Ind.).

Jung, H. Ein Beitrag zur nichtlinearen Elastizitätstheorie. Ing.-Arch. 21, 194–207 (1953).

The theory proposed by the author is identical with that of Gleyzal [J. Appl. Mech. 13, A-261-A264 (1946); these Rev. 9, 120], whose position with respect to other quasi-linear theories, as well as the author's various manipulations, have been presented by the reviewer [J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); 2, 593–616 (1953), §51 and §9; these Rev. 13, 794, and the second preceding review]. In §§6–8 the author discusses exact solutions for homogeneous strain and for torsion; since in the author's theory there is no strain energy, these solutions are not included in those of Rivlin [see §42 of the reviewer's paper cited above]. The equations governing plane strain and a loaded strip of plating are developed. *C. Truesdell.*

Green, A. E., and Wilkes, E. W. A note on the finite extension and torsion of a circular cylinder of compressible elastic isotropic material. Quart. J. Mech. Appl. Math. 6, 240–249 (1953).

The authors study the simultaneous finite extension and torsion of a circular cylindrical rod of compressible perfectly elastic material, using a method of power series expansion in the twist, ψ. As far as terms of order ψ^2, their results have been obtained more simply and in greater generality by Rivlin in work done later but published earlier [J. Rational Mech. Anal. 2, 53–81 (1953), §§5–6; these Rev. 14, 513]. The authors obtain also the twisting moment correction of order ψ^2. For the case when the plane ends are free, their solution shows that there is in general a change of radius of order ψ^2, while the special stress-strain relations used by Seth [Philos. Trans. Roy. Soc. London. Ser. A. 234, 231–264 (1935)] yield a change only of order ψ^4. [This observation on Seth's theory is essentially equivalent to that made by the reviewer in connection with simple shear [J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); §53; these Rev. 13, 794; and the paper reviewed third above], and hence it will apply equally to Signorini's most recent theory [summarized in Ann. Mat. Pura Appl. (4) 30, 1–72 (1949); these Rev. 11, 756].] *C. Truesdell.*

Sbrana, Francesco. Una proprietà caratteristica delle equazioni dell'elasticità. Atti Accad. Ligure 9 (1952), 84–88 (1953).

By using Tedone's identity and the mean-value theorem for harmonic functions, the author obtains a simple and elegant proof of the following mean value theorem for classical static elasticity theory. Let Σ be a sphere of radius R about P, let Q lie on Σ, let Rn_i be the vector from P to Q. Put

$$H_{ij}=\frac{3}{6+2q}[(2-q)\delta_{ij}+5qn_in_j].$$

Then in order that, in a neighborhood about P, s_i shall satisfy Navier's statical equation for an isotropic body in which $q=1+\lambda/\mu$, it is necessary and sufficient that

$$4\pi R^2 s_i=\oint_\Sigma H_{ij}s_j dS$$

for all spheres Σ in the neighborhood. [The necessity of a similar formula with a different tensor H_{ij} was recently proved by Aquaro [Rivista Mat. Univ. Parma 1, 419–424 (1950); Rend. Sem. Fac. Sci. Univ. Cagliari 21, 43–46 (1952); these Rev. 12, 770; 14, 221]; it would be interesting to work out the connection between these results.] *C. Truesdell* (Bloomington, Ind.).

***Grioli, Giuseppe. Proprietà di media ed equilibrio elastico.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 68–77. Casa Editrice Perrella, Roma, 1953.

Lecture on material more fully developed in other papers [Ann. Mat. Pura. Appl. (4) 33, 239–246, 263–271 (1952); these Rev. 14, 514, 515.] *C. Truesdell.*

Bland, D. R. Mean displacements on the boundary of an elastic solid. Quart. J. Mech. Appl. Math. 6, 379–384 (1953).

The author derives Betti's formula for the volume average of the strain tensor in an elastic body without using Betti's reciprocal theorem and observes that this average can also be expressed in terms of a surface integral of the displacement vector. Though he treats only isotropic materials, a similar derivation can be given for anisotropic materials. Problems of plane strain and compression of a circular cylinder are discussed. *J. L. Ericksen.*

Karunes, B. On the concentration of stress round the edge of a hole bounded by two intersecting circles in a large plate. Indian J. Phys. 27, 208–212 (1953).

A stress function in bi-polar coordinates has been obtained to give the distribution of stress around the edge of a hole bounded by two intersecting circles in an infinite plate under uniform shear in the plane of the plate; and some particular cases have been discussed. (Author's abstract.) *R. M. Morris* (Cardiff).

Chong, Frederick. Solution by dual integral equations of a plane-strain Boussinesq problem for an orthotropic medium. Iowa State Coll. J. Sci. 27, 321–334 (1953).

It is first shown that provided the coefficients of a given polynomial $u^1g(u)$ are chosen to satisfy a certain relation, then a solution for $f(t)$ of the pair of dual integral equations

$$\int_0^\infty t^p f(t)J_p(ut)dt=g(u), \quad 0<u<1$$

$$\int_0^\infty f(t)J_p(ut)dt=0, \quad u>1$$

can be found even when the condition

$$s>-2, \quad -p-1<s-\tfrac{1}{2}<p+1,$$

laid down by Busbridge [Proc. London Math. Soc. (2) 44, 115–129 (1938)] is violated.

This particular solution is then applied to the problem of a semi-infinite orthotropic elastic medium having its normal along the x-axis, which is such that its physical properties are unaltered when the axes y and z are interchanged, and which is indented by a rigid punch in the form of an infinite cylinder with its axis parallel to the z-axis. The form of the

indentation being given as a polynomial

$$(*) \qquad x=g(y)=\sum_{n=0}^{m} A_n|y|^n, \quad |y|<a$$

the problem is a two-dimensional one of plane-strain. The particular case $g(y)=$const., is first solved as a special case which requires independent treatment, and this solution is also required for the general case given by (*). The stress components in the medium are found in terms of elementary integrals and the results for the isotropic case can be deduced. These are said to be in agreement with those of other writers. *R. M. Morris* (Cardiff).

Žgenti, V. S. Application of functional analysis in the theory of bent thin elastic shells of small curvature. Doklady Akad. Nauk SSSR (N.S.) **91**, 217–219 (1953). (Russian)

Let the middle surface of the bent shell be the surface $z=f(x, y)$, where f has continuous second partial derivatives in a domain G_0 with boundary L_0, and write, in symbolic form, the equations of equilibrium of the shell in the form $AU=q$, which is a system of three second-order partial differential equations for the displacement vector $U=(u, v, w)$ of the shell. The boundary-value problem considered consists of this system of partial differential equations in G_0, plus the boundary conditions $u=v=w=\partial w/\partial \nu=0$ on L_0. The existence of a solution is shown by proving that, in a suitable Hilbert space whose elements are "displacement vectors" satisfying the boundary conditions, the operator A is positive definite. An imbedding theorem of S. L. Sobolev [Certain applications of functional analysis in mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; these Rev. **14**, 565] guarantees the uniform convergence, in G_0+L_0, of a minimizing sequence of the equivalent variational problem to the solution of the differential problem. *J. B. Diaz* (College Park, Md.).

Paria, Gunadhar. Stresses in a thin elastic plate with a parabolic boundary due to normal pressures distributed near the vertex. Bull. Calcutta Math. Soc. **44**, 181–183 (1952).

Šeremet'ev, M. P. Bending of thin plates with reinforced boundaries. Ukrain. Mat. Žurnal **5**, 58–79 (1953). (Russian)

The paper gives a formulation of the problem of small transverse deflections of a thin elastic plate reinforced along the edge by a ring whose thickness and rigidity differ from those of the plate. The middle surfaces of the plate and the ring, in the undeformed state, lie in the (x, y)-plane and the cylindrical edge of the plate is welded onto the ring. The plate so reinforced is then deformed by forces and moments distributed along its edge. The state of stress in the interior of the homogeneous and isotropic plate is thus determined by two analytic functions $\varphi_i(z)$ of the complex variable $z=x+iy$, which satisfy certain conditions along the contour of the weld depending on the nature of the reinforcing ring. If the ring is so thin that it behaves like an inextensible curved rod in the Kirchhoff-Clebsch theory of thin rods, it is possible to deduce the appropriate boundary conditions. The author calculates the functions $\varphi_i(z)$ for an infinite plate with a circular hole when the plate is deformed by the application of constant bending and twisting moments at a great distance from the hole and when the ring is free of external loads. *I. S. Sokolnikoff.*

de Schwarz, M. J. Über das Verhalten der Torsionsfunktion in der Nähe von einspringenden Ecken massiver und hohler Stäbe. Österreich. Ing.-Arch. **7**, 88–100 (1953).

It is known from the St. Venant theory of torsion that infinite stresses exist at reëntrant corners and are locally of the form $(\gamma/k)\rho^{(1-k)/k}\sin(\theta/k)$, where (ρ, θ) are polar coordinates at the apex of the angle $k\pi$ $(k>1)$. The question arises, is $\gamma\neq 0$? The author shows $\gamma<0$ if the reëntrant corner is on an inner boundary, and $\gamma>0$ if on an outer boundary of a multiply connected region. For a simply connected region $\gamma>0$. The proof is given by establishing bounds for the torsion function. A basic theorem by Picone concerning the line integral of a normal derivative is employed in the analysis. An example is given for a hollow cross-section bounded by two equal and parallel-sided regular polygons. *D. L. Holl* (Ames, Iowa).

Ōkubo, Hajimu. Torsion of a circular shaft with diameter varying periodically along its length. Z. Angew. Math. Physik **4**, 197–207 (1953).

In the torsion problem of a circular shaft of variable diameter the stress function $\psi(r, z)$ satisfies a second-order partial differential equation and a boundary condition $\psi=$const. for a simply connected region. The author studies a shaft whose diameter varies periodically along the z-axis, in particular a screw-threaded shaft, whose boundary is not given by a single function $r=f(z)$. For the approximations employed in the general form of a periodic solution, it is necessary to solve infinite linear systems of equations in order to satisfy the boundary requirement. In the example, the author calculates the error in the normal shearing stress at the boundary. It is also stated that the stiffness of the threaded shaft is slightly larger than that of a shaft without grooves, but having a diameter equal to the minimum diameter of the threaded shaft. *D. L. Holl.*

Luxenberg, Harold. Torsion of anisotropic elastic cylinders by forces applied on the lateral surface. J. Research Nat. Bur. Standards **50**, 263–276 (1953).

The St. Venant theory solves the torsion problem of an isotropic or anisotropic rod of uniform section for the relaxed end conditions of pure couples and with the lateral surface free of stress. This paper considers the anisotropic rod (13 elastic constants) which is clamped at one end and is twisted by forces applied uniformly along the lateral surface. The problem is attacked by assuming the final stress system is the sum of three systems, a plane strain system, a modified torsion system for lateral surface forces and relaxed conditions at the clamped end and a linear stress system which requires the resulting bending couples and tension to vanish on the end-sections. The equilibrium equations and compatibility relations require that the plane stress function $\psi(x, y)$ and the modified torsion function $\phi(x, y)$ simultaneously satisfy two partial differential equations involving second-order derivatives of ϕ and fourth-order derivatives of ψ. However, the boundary conditions are such that the boundary-value problems are separable, and the torsion problem is reducible to a Neumann problem. With this problem completed the stress function ψ satisfies a soluble fourth-order equation. The author shows the degenerate case of an isotropic rod and also solves the case of an anisotropic elliptic rod where the results become polynomials in x, y. *D. L. Holl* (Ames, Iowa).

Nakazawa, Hajime. Torsion of a shaft with a number of longitudinal semi-circular notches. Mem. Fac. Tech. Tokyo. Metro. Univ. **1953**, 117–126 (1953).

The object of this paper is to solve the St. Venant torsion problem of a splined shaft whose cross-section is a circle with notches on its boundary. The treatment is more general in that the analysis allows curvilinear notches resembling a semi-circle. Moreover, the center of curvature at the greatest depth of the notch is not necessarily on the boundary of the base circle. The solution is accomplished by using curvilinear coordinates characterized by

$$z^m=(x+iy)^m=a^m\tan\tfrac{1}{2}(\alpha+i\beta),\quad 2m=2,3,4,5,\cdots.$$

The circle of radius a corresponds to $\alpha=\pm\pi/2$. On this circle are $2m$ points corresponding to $\beta=\infty$. Closed curves corresponding to $\beta=\beta_1$ surround these points. The problem is that of determining ψ harmonic interior to the region bounded by $\alpha=\pm\pi/2$ and $\beta=\beta_1$ and attaining the value $(x^2+y^2)/2$ on this boundary. The author shows the maximum boundary stress on a semi-circular notch is double the usual value (no notches) and is independent of the number of notches. Expressions for the moment and stresses are provided. The case $m=2$ (four notches) is shown graphically.

D. L. Holl (Ames, Iowa).

Gray, C. A. M. An iterative solution to the effects of concentrated loads applied to long rectangular beams. Quart. Appl. Math. **11**, 263–271 (1953).

The problem of stress distribution in thin deep rectangular beams subjected to the action of concentrated loads is of great practical interest and has received considerable attention in the past. The beam has been assumed to be infinitely long and to be in a state of generalised plane stress, and in previous methods the stresses and displacements have been found in integral form which have been difficult to interpret. The method of the author is to transform the infinite strip representing the beam into the interior of a circle and then, using Mushelišvili's two-dimensional complex stress functions and the required boundary conditions, the solution is obtained in the form of a complex power series, the coefficients of which are determined by an iterative process. The displacement and curvature of the centre line are determined, and a graphical illustration shows that at a distance from the boundary greater than half the depth of the beam the curvature is proportional to the bending moment. The reviewer would have liked a little more explanation of how the load function was obtained.

R. M. Morris (Cardiff).

Nardo, S. V. An exact solution for the buckling load of flat sandwich panels with loaded edges clamped. J. Aeronaut. Sci. **20**, 605–612 (1953).

Diese Arbeit gibt eine exakte Lösung für die Beullast einer geschichteten ebenen Rechteckplatte, deren belastete Ränder eingespannt, während deren unbelastete Ränder einfach unterstützt sind. Die Theorie ist auch anwendbar für den Fall, dass die Randbedingungen der belasteten und unbelasteten Ränder vertauscht werden. Die Lösung hat ihre Grundlage in der Plattentheorie kleiner Ausbiegungen, wobei angenommen wird, dass die Spannungen immer unterhalb der Proportionalitätsgrenze sind. Das Material sowohl der Aussenhaut als des Innenkerns der geschichteten Platte wird isotrop angenommen. Drei Bedingungen des Gleichgewichts, abgeleitet mit Hilfe des Prinzips der virtuellen Verschiebungen, bilden den Ausgangspunkt der Theorie. Diese Gleichungen sind partielle Differentialgleichungen mit konstanten Koeffizienten, deren zwei von der zweiten Ordnung, während die dritte Gleichung vierter Ordnung ist. Sowohl diese Gleichungen wie sechzehn Randbedingungen sind identisch befriedigt. Für den Fall belasteter und eingespannter Ränder sind numerische Berechnungen durchgeführt. Eine Reihe berechneter Diagramme stellen die Beullast dar als Funktion des Seitenverhältnisses der Platte sowie des Verhältnisses der Kerndicke zur Plattendicke. Jede Kurve ist gültig für den besonderen Wert eines Parameters, der das Verhältnis der Schersteifigkeit zur Biegesteifigkeit des Kerns bzw. der Platte darstellt. Werte dieses lezteren Parameters sind so gewählt, dass sie das Gebiet der meisten praktisch vorkommenden Fälle decken. Eine Überprüfung der Theorie durch Versuche am Forest Products Laboratory zeigen eine vernünftige Übereinstimmung zwischen Theorie und experimentellen Ergebnissen.

R. Gran Olsson (Trondheim).

Ledinegg, E., und Urban, P. Zur theorie der Eigenschwingungen isotroper elastischer Medien. Acta Physica Austriaca **7**, 420–435 (1953).

The authors use a perturbation method to treat problems involving small vibrations of an inhomogeneous, isotropic elastic medium, assuming that the elastic coefficients and density are nearly constant. A considerable part of their analysis is devoted to the case where the displacement vanishes over the boundary of a region. (Results obtained using this assumption are trivial, for the uniqueness theorems of classical elasticity then show that the displacement vanishes identically.) They do show that, in the case of free vibrations, the change in frequency of the normal modes of vibration resulting from replacing constant by variable elastic coefficients can be expressed in terms of the solution for the homogeneous material. *J. L. Ericksen.*

Toriumi, Isao. Natural frequencies of axial-symmetrical vibration of finite elastic body with additional mass on the surface. Tech. Rep. Osaka Univ. **3**, nos. 55–73, 125–129 (1953).

***Draminsky, Per. Étude sur les sous-harmoniques dans les vibrations de torsion de vilebrequins et remarques générales sur les oscillations sous-harmoniques dans les systèmes non linéaires. Subharmonics in torsional crankshaft vibrations and some general remarks about subharmonics in non-linear systems.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 129–157; discussion, p. 158, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and English)

Mencher, Alan G. Epicentral displacement caused by elastic waves in an infinite slab. J. Appl. Phys. **24**, 1240–1246 (1953).

The author attempts to obtain the solution of the problem using the Laplace transforms of the scalar and vector potentials. The scalar potential is expressed as the sum of a term representing the action of a source of longitudinal waves located centrally, the other term representing a "perturbation caused by the presence of the boundaries" and terms corresponding to an infinite set of images. These terms are, of course, also due to the presence of the boundaries but are not determined by some coefficients of reflection. The constants in these expressions are determined from the boundary conditions in the usual manner. The resulting integral for the displacement is transformed by expanding

The reviewer apologizes to the authors for making the erroneous statement beginning in line 6.

two factors in the integrand in series. Each of the new integrals should represent one of the reflected waves. The first four terms are evaluated and three of them diverge as time increases. The author expects to give at a later date a rigorous proof of the convergence of series used. A justification of the method is now seen in the fact that the displacement represented by the integral before the series expansion is finite as $t\to\infty$. *W. S. Jardetzky* (New York, N. Y.).

***Kuessner, H. O. On the mathematical treatment of movements of the earth's crust.** The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1953. i+20 pp. (mimeographed).

All the movements of the Earth's crust are reduced in the author's treatment to the floating of an elastic plate. For the discussion of such a floating plate he makes use of the E. Reissner's theory of plate deformations [Quart. Appl. Math. **5**, 55–68 (1947); these Rev. **8**, 547], assuming that a load is given on the upper surface of the plate. At the lower surface this plate is supported by a viscous fluid. The equations are linearized and solutions for small deflections expressed in terms of different functions. According to the author these solutions could yield an explanation of a "graben" or of ring mountains as well as of "folded" mountains. No references are made to other investigations, not even to the work of H. Jeffreys, "The earth" [3rd ed., Cambridge, 1952], where similar problems are also discussed. *W. S. Jardetzky* (New York, N. Y.).

Reuss, E. Die Stoffgleichung hochviskoser Flüssigkeiten und ihre Anwendung auf den Ultraschall. Acta Tech. Acad. Sci. Hungar. **6**, 65–78 (1953). (Russian, English and French summaries)

The author proposes to use the Maxwellian relaxation theory as extended by Natanson to describe ultrasonic absorption and dispersion. He obtains and discusses the exact absorptions and dispersion measures for longitudinal and transversal waves. In his summary he states that "conformity with test results is not satisfactory, but the suggested equation forms a logically exact basis to dispose of discrepancies." [Other attempts to treat absorption and dispersion by a theory of visco-elasticity are summarized in §4 of the reviewer's paper in J. Rational Mech. Anal. **2**, 643–741 (1953).] *C. Truesdell* (Bloomington, Ind.).

Graffi, Dario. Sulla teoria dei materiali elastico-viscosi. Atti Accad. Ligure **9** (1952), 77–83 (1953).

The author first discusses the superposition theories of visco-elasticity, in which, for the case of infinitesimal displacements and strains e_{ij}, the constitutive equations are

$$(1)\qquad Pt_{ij}=Q[\lambda e_{kk}\delta_{ij}+2\mu e_{ij}],$$

where t_{ij} is the stress tensor and where P and Q are polynomials of degrees m and n in $\partial/\partial t$ with constant coefficients. Throughout the analysis, the inertial terms in the dynamical equations are neglected. The author's objective is to connect the theorem of Alfrey and Tsien [Tsien, Quart. Appl. Math. **8**, 104–106 (1950); these Rev. **11**, 628] with an analogous result of his own in the Boltzmann-Volterra accumulative theory [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 25–30 (1951); these Rev. **13**, 183]. These results state, roughly, that for a stress boundary-value problem the stresses will be exactly the same as for the same problem set within the classical linear theory of elasticity, although of course the displacements will in general be quite different. The author begins by applying the Laplace transformation to a result derived from (1). When $m=n$, it turns out that the visco-elastic theory is equivalent to a special case of the special case of the Boltzmann-Volterra theory for which the author's theorem holds; thus, when $m=n$, the Alfrey-Tsien theorem follows from his. When $m\neq n$, however, the same transformation leads to a different type of accumulative theory. This type is included in a new class which the author now proposes for study:

$$(2)\qquad e_{ij}(t)=\theta[\chi_{ij}(\tau)]$$

where $\chi_{ij}\equiv(1+\sigma)E^{-1}t_{kk}\delta_{ij}-\sigma E^{-1}t_{ij}$, and where θ is a single-valued linear functional of its argument in $0\leq\tau\leq t$. To this more general class of substances the author extends his earlier theorem, thus including all cases of the Alfrey-Tsien theorem. The argument is the same as before. Let $t_{ij}^{(0)}$, $e_{ij}^{(0)}$ be the solution from elasticity theory. Then Cauchy's dynamical equations are satisfied. Define e_{ij} by (2) with $t_{ij}\equiv t_{ij}^{(0)}$; then the author shows that since $e_{ij}^{(0)}$ satisfies St. Venant's conditions of compatibility it follows from (2) that e_{ij} satisfies them also. Hence the system $t_{ij}^{(0)}$, e_{ij} satisfies all requirements of the new theory. Consequently, whenever uniqueness holds in the new theory, no other stress than $t_{ij}^{(0)}$ is possible. *C. Truesdell* (Bloomington, Ind.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Gorn, Saul. Series expansions of rays in isotropic, non-homogeneous media. Quart. Appl. Math. **11**, 355–360 (1953).

The author discusses the propagation of rays in inhomogeneous isotropic media, using matrix and vector methods at the same time. He computes curvature and torsion of light rays as a function of the refractive index and gives recursion formulae for the expansion of the formulae for curved rays into a power series. *M. Herzberger.*

Bartkowska, J. Third order aberrations of a mirror lens. Acta Phys. Polonica **12**, 57–63 (1953). (Russian summary)

The author performs the very useful job of transforming Seidel's aberration formulae to the practically important case of systems containing refracting and reflecting surfaces, especially the case of a mirror lens, i.e., a single lens with reflecting second surface. *M. Herzberger.*

v. Fragstein, C., und Schaefer, Cl. Zur Strahlversetzung bei Reflexion (Erwiderung an Herrn Artmann). Ann. Physik (6) **12**, 84–88 (1953).

The authors discuss and refute arguments made by Artmann [Ann. Physik (6) **7**, 209–212 (1950); **8**, 285–290 (1951); these Rev. **13**, 707] against former papers of the authors in which they discussed the shifting of rays for the phenomenon of total reflection with the help of the Poynting vector. They point out, that they have, in experiment and in theory, seen to it that the approximations made remain valid within the range considered and that, for instance, diffraction may be neglected. *M. Herzberger.*

Müller, Rolf, und Westpfahl, Konradin. Eine strenge Behandlung der Beugung elektromagnetischer Wellen am Spalt. Z. Physik **134**, 245–263 (1953).

The authors compute the electromagnetic scattering from a slit by a method used by Bouwkamp in the case of the scalar wave diffraction problem for the disk [Physica **16**, 1–16 (1950); Philips Research Rep. **5**, 401–422 (1950); these Rev. **11**, 755; **12**, 774]. This work has been duplicated by Bouwkamp in a recent report [Mathematics Research Group, New York University, Research Rep. No. **EM-50** (1953); these Rev. **14**, 1148]. *W. K. Saunders.*

Hönl, H., und Zimmer, Eva. Intensität und Polarisation bei der Beugung elektromagnetischer Wellen am Spalt. II. Z. Physik **135**, 196–218 (1953).

In previous papers [Hönl, Z. Physik **131**, 290–304 (1952); these Rev. **13**, 801; Groschwitz and Hönl, ibid. **131**, 305–319 (1952); these Rev. **13**, 890] the problem of the diffraction of a field of plane electromagnetic waves on an infinitely long slit of width $2b$ was treated for an incident electric vector parallel to the boundaries of the slit. Analogous investigations are carried out here for a perpendicular incident electric vector, corresponding to a magnetic vector parallel to the boundaries of the slit. The combination of both cases solves the problems of an incident electric vector with arbitrary polarization. Again the problem is reduced to the problem of the solution of a dual integral equation

$$\text{(a)} \qquad \int_{-\infty}^{\infty} \gamma(\alpha)h(\alpha)e^{i\alpha\xi}d\alpha = 0, \quad |\xi| > b,$$

$$\text{(b)} \qquad \int_{-\infty}^{\infty} h(\alpha)e^{i\alpha\xi}d\alpha = 1, \quad |\xi| < b,$$

where $\gamma(\alpha) = (1-\alpha^2)^{1/2}$ for $|\alpha| \leqq 1$ and $i(\alpha^2-1)^{1/2}$ for $|\alpha| > 1$. For the solution of (a), (b) the attempt

$$\text{(c)} \qquad \gamma(\alpha)h(\alpha) = \sum_{m=0}^{\infty} B_m J_{2m}(kb\alpha)$$

is made which satisfies (a) by the Weber-Sonine integral formula. The B_m have to be determined such that (c) satisfies (b). Thus an infinite system of linear equations with B_m as the unknown is derived which admits a successive solution. For the case of a narrow slit ($kb \ll 1$) the first two coefficients in (c) are evaluated. The intensity and the polarization of the diffracted field in case of (1) a polarized and (2) an unpolarized incident field are thoroughly discussed and illustrated by numerical evaluations and graphical representations.

The connection between the method used here to that employed by Müller and Westphal [see the paper reviewed above] for the solution of the same problem is shown in an appendix. *F. Oberhettinger* (Washington, D. C.).

Hönl, H., und Maue, A.-W. Die Eindeutigkeit der Lösungen in der strengen Beugungstheorie. Z. Physik **132**, 569–578 (1 plate) (1952).

In his paper in Z. Physik **131**, 290–304 (1952) [these Rev. **13**, 801] the first author presented the solution to the plane diffraction problem for the scalar wave in terms of two Fourier-type integral equations. In the present paper it is shown that terms of the product solution type in cylindrical coordinates which violate the Meixner edge condition [Ann. Physik (6) **6**, 2–9 (1949); these Rev. **11**, 562] are not solutions of the homogeneous parts of these integral equations. The extension to curved screen boundaries and to questions of uniqueness is discussed. *W. K. Saunders.*

Poincelot, Paul. Sur l'inexistence de l'onde de surface (Oberflaechenwelle) de Sommerfeld. Ann. Télécommun. **8**, 206–211 (1953).

The author examines in detail a crucial part of Sommerfeld's celebrated paper on radio wave propagation over plane earth [Sommerfeld, Ann. Physik (4) **28**, 665–736 (1909)]. He shows that Sommerfeld erred in evaluating asymptotically a certain contour integral around a branch cut, so that Sommerfeld's theory of the surface wave must be rejected. Many other objections to Sommerfeld's analysis, that can be made according to the reviewer, are left unmentioned. A rather complete bibliography on the subject is included. *C. J. Bouwkamp* (Eindhoven).

Deryugin, L. N. Equations for reflection coefficients of waves from a periodically uneven surface. Doklady Akad. Nauk SSSR (N.S.) **87**, 913–916 (1952). (Russian)

The perfectly conducting surface has the form of a plane furrowed by rectangular grooves at regular intervals. The incident plane wave has magnetic vector parallel to the grooves. The proposed method uses two infinite series of plane waves to represent the fields inside and outside the grooves. Continuity conditions then yield infinite systems of linear equations for the unknown coefficients, to be handled by approximate methods. The dependence of reflection coefficients is investigated numerically for grooves of depth up to half a wave-length. *F. V. Atkinson.*

Montroll, Elliott W., and Greenberg, J. Mayo. Scattering of plane waves by soft obstacles. III. Scattering by obstacles with spherical and circular cylindrical symmetry. Physical Rev. (2) **86**, 889–898 (1952).

[For parts I and II see Montroll and Hart, J. Appl. Phys. **22**, 376–386, 1278–1289 (1951); these Rev. **13**, 94; **14**, 110.] The authors consider the scattering problem for the equation

$$\nabla^2\psi + k^2(r)\psi = 0$$

where $\int_0^\infty [k^2(r) - k_0^2]r^2dr$ and $\int_0^\infty [k^2(r) - k_0^2]rdr$ are bounded. They replace the exact problem by the approximate problem

$$\nabla^2\psi_0 + k_0^2(r)\psi_0 = 0$$

where

$$k_0^2(r) - k_0^2 = \begin{cases} k_1^2 - k_0^2, & r < a \\ 0, & r > a. \end{cases}$$

They then determine parameters a and k_1^2 by means of the variational requirement $\delta\int |k^2(r) - k_0^2(r)|\psi_0^2 dr = 0$. For ψ_0 they use the approximate solution valid when k^2 does not differ markedly from k_0^2 (high energies in quantum mechanics) which they derived in an earlier paper. Explicit equations are found for a, $k_1^2 - k_0^2$, the angular distribution, and the total cross-section. They apply these methods to the case of the Gaussian exponential and Yukawa potential. *H. Feshbach* (Cambridge, Mass.).

Lien, Roy Harold. Radiation from a horizontal dipole in a semi-infinite dissipative medium. J. Appl. Phys. **24**, 1–4 (1953).

An electric dipole with its axis parallel to the x-axis is placed at a point $(0, 0, h)$. The medium underneath the (x, y)-plane ($z < 0$) is considered to be dissipative while that above ($z > 0$) is air. [See Sommerfeld, Partial differential equations in physics, Academic Press, New York, 1949, pp. 257–265; these Rev. **10**, 608.] The electric field components can be represented in the form of definite integrals involving Bessel functions. It is shown here that approxi-

mate expressions for all the occurring integrals can be given if the frequency is low. The formulas thus obtained are used to give expressions for the electric field components inside the region $z<0$. *F. Oberhettinger* (Washington, D. C.).

Dufour, Charles, et Herpin, André. Propagation des ondes électromagnétiques dans un milieu stratifié périodique transparent. Rev. Optique **32**, 321–348 (1953).

The problem of the title is analyzed by the use of Pauli spin matrices, for the case of normally incident electromagnetic waves and piles of double layers of equal optical thickness. Various characteristics of certain numerical examples are displayed in a number of graphs.

C. J. Bouwkamp (Eindhoven).

***Caprioli, Luigi. Onde E. M. di tipo trasversale nelle guide d'onda rettilinee e con dielettrico eterogeneo.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 478–483. Casa Editrice Perrella, Roma, 1953.

It is shown that a rectangular wave guide filled with a dielectric material which is a function of the coordinates x, y, z cannot have a TEM mode. *A. E. Heins.*

Specht, Wilhelm. Eine mathematische Frage der Strahlentherapie. J. Reine Angew. Math. **191**, 92–96 (1953).

A bounded closed region P in space is filled with a radiating substance. The radiation density at a point of P specified by the position vector $\mathbf{p}$ is $\sigma(\mathbf{p})$ (>0), so that the total radiation is represented by the integral $\int_P \sigma(\mathbf{p})d\mathbf{p}$, which can, by suitable normalisation, be taken to be unity. It is assumed that at a point $\mathbf{a}$ outside P the intensity of radiation is given by the expression

$$I(\mathbf{a})=\int_P \frac{\sigma(\mathbf{p})d\mathbf{p}}{|\mathbf{a}-\mathbf{p}|^2}.$$

In practice $I(\mathbf{a})$ is not observed directly. The quantity which is measured is the total intensity inside an ionisation chamber, which is assumed here to be a sphere with centre $\mathbf{a}$. If the radius of this sphere is taken as unit of length, the mean intensity of radiation in the ionisation chamber is

$$\bar{I}(\mathbf{a})=\frac{3}{4\pi}\int_K\int_P \frac{\sigma(\mathbf{p})}{|\mathbf{a}-\mathbf{r}-\mathbf{p}|^2}d\mathbf{p}d\mathbf{r}$$

where K is the unit sphere $|\mathbf{r}|\leq 1$. It is usual (e.g., in radiotherapy) to take $\bar{I}(\mathbf{a})$ as a measure of $I(\mathbf{a})$. This is satisfactory so long as the distance from the radiating material in P is large compared with the radius of the ionisation chamber ($|\mathbf{a}|\gg 1$). The object of this paper is to find an estimate of the relative error $[\bar{I}(\mathbf{a})-I(\mathbf{a})]/I(\mathbf{a})$ independent of the density $\sigma(\mathbf{p})$ and the form of P. *E. T. Copson.*

Piefke, Gerhard. Die "Exponentialleitung" und ihre Wellenablösung. I, II. Arch. Elektr. Übertragung **7**, 229–235, 274–280 (1953).

It is first demonstrated that the so-called "exponential" transmission line, for which the capacitance is proportional to $e^{2\beta z}$ and the inductance proportional to $e^{-2\beta z}$ with the propagation constant $\gamma=\pm j\alpha+\beta$ cannot be realized. The author then reverses the problem and starting with the assumed exponential variation of capacitance, finds for the plane case the needed variation of distance between the planes given by $a(z)=(1/2\beta)\sin^{-1}(e^{-2\beta z})$. The further solution of this transmission line problem is reduced to the second order differential equation

$$\frac{d^2y}{dz^2}+g(z)\frac{dy}{dz}+h(z)=0 \tag{1}$$

for which a new method of solution is proposed. Introducing the new variable $u(z)=C\int \exp\left[-\int g(z)dz\right]dz$ into (1) eventually leads to the form

$$\frac{d^2y}{du^2}+f(u)y=0. \tag{2}$$

The assumption $y=A\exp\left(\int_{u_0}^{u}\gamma du\right)$ gives then for the propagation constant γ the Riccati-type equation

$$\frac{d\gamma}{du}+\gamma^2+f(u)=0. \tag{3}$$

Expanding here $f(u)=\sum_{n=0}^{\infty}a_n/u^n$, one can get for γ by coefficient comparison the infinite series $\gamma=\sum_{n=0}^{\infty}\gamma_n e^{-n\beta z}$ where the γ_n are determined from a recursion formula. The first term of this series is the same one that had been used in the former incomplete theories of the "exponential" line; however, the convergence of the series is acceptable only for lower frequencies close to the cut-off wavelength which is defined by $\lambda_c=2\pi/\beta$. The line has the character of a high pass filter.

The same procedure for the inhomogeneous coaxial cylinder line leads to a highly transcendental equation for the conductor shapes involving Bessel and Neumann functions.

Since the conventional transmission line equations disregard the longitudinal component of the electric field, the author also considers the complete solution of the Maxwell equations for the plane case and finds that for higher frequencies the field pattern is very much like the radiation field in the antenna problem compressed between the conducting surfaces. *E. Weber* (Brooklyn, N. Y.).

Lettowsky, Felix. Eine Methode zur Berechnung des Hochfrequenzwiderstandes zylindrischer Leiter allgemeiner Querschnittsform. Arch. Elektrotechnik **41**, 64–72 (1953).

In order to solve the skin-effect problem for any general cross-section of cylindrical conductors the author uses a conformal transformation of the given cross-section in the z-plane into a circle in the z-plane and solves the two-dimensional diffusion equation for the vector potential A in the z-plane by the integral

$$A=K_0\int_0^{2\pi}\psi_0(\lambda)\left[1+\frac{f_1(\xi_1,\lambda)}{i\alpha}+\frac{f_2(\xi_1,\lambda)}{(i\alpha)^2}+\cdots\right]\times\exp\{-i\alpha[W(\lambda)-(x\cos\lambda+y\sin\lambda)]\}d\lambda \tag{1}$$

where $\xi=\xi_1$ defines the conductor surface, and the functions $W(\lambda)$, $\psi_0(\lambda)$, $f_1(\xi_1,\lambda)$ and $f_2(\xi_1,\lambda)$ are directly determined from the boundary conditions. Actually, the integral in (1) is evaluated by asymptotic expansion about a point λ_0 whereby λ_0 usually but not necessarily is chosen as the saddle point of the integrand in (1). The explicit asymptotic forms of A and of the characteristic ratios of a-c resistance and a-c reactance to d-c resistance are deduced. Reference is made to an earlier solution for the elliptic cross-section [Arch. Elecktrotechnik **35**, 643–662 (1941)].

Specific application is made to two parallel wires (Lecher wires) and it is demonstrated that for large distance between the wires, the expansion becomes identical with Sommerfeld's result for the single round wire. Application to

rectangular cross-section is indicated and promised in a later publication. *E. Weber* (Brooklyn, N. Y.).

*Finzi, Bruno. **Sul principio della minima azione e sulle equazioni elettromagnetiche che se ne deducono.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 501–506. Casa Editrice Perrella, Roma, 1953.

This paper deals with Maxwell's equations in flat space-time and their generalisations. In deriving Maxwell's equations for the electromagnetic tensor $F_{\alpha\beta}$ from a variational principle, it is usual to assume one half of these equations, so that $F_{\alpha\beta}=\Phi_{\beta|\alpha}-\Phi_{\alpha|\beta}$ where Φ_α is the potential 4-vector, and then obtain the other half from $\delta\int l d\omega=0$ using the Lagrangian $l=\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}-\Phi_\alpha j^\alpha$ where j^α is the 4-current ($d\omega$ is the element of 4-volume). In this paper the author does not assume any part of Maxwell's equations, but derives everything from a variational principle. He assumes, as a generalisation of a well known decomposition of a vector in 3-space, that any skew-symmetric tensor $F_{\alpha\beta}$ may be written $F_{\alpha\beta}=H_{\alpha\beta}+K_{\alpha\beta}$ where $H_{\alpha\beta}$ is irrotational and $K_{\alpha\beta}$ solenoidal in the sense that they satisfy rot $H_{\alpha\beta}=0$, div $K_{\alpha\beta}=0$, where rot and div are defined by rot $H_{\alpha\beta}=\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}H^{\alpha\beta|\gamma}$, div $K_{\alpha\beta}=K^{\beta}_{\alpha\beta}$. [To the reviewer this notation appears unsatisfactory; since the rot and div are vectors, it would be better to use symbols such as $\mathrm{rot}_\beta H$, $\mathrm{div}_\alpha K$. This is a criticism of notation only; the meaning is clear.] Whether this decomposition of $F_{\alpha\beta}$ is possible in general or not (no attempt is made to prove it), the fact is that the author considers electromagnetic tensors $F_{\alpha\beta}$ which are expressible in terms of two vectors Φ_α, Ψ_α by

$$F_{\alpha\beta}=H_{\alpha\beta}+K_{\alpha\beta},\quad H_{\alpha\beta}=\Phi_{\beta|\alpha}-\Phi_{\alpha|\beta},\quad K_{\alpha\beta}=\epsilon_{\alpha\beta\gamma\rho}\Psi^{\gamma|\rho}.$$

Then the whole set of Maxwell's equations (without current) follow from $\delta\int l d\omega=0$ with $l=\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ and arbitrary variations $\delta\Phi_\alpha$, $\delta\Psi_\alpha$. Generalisations are obtained by taking for l other quadratic invariants formed from Φ_α, Ψ_α and their first derivatives, together with j_α. In particular, the Lagrangian

$$l=\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}-j_\alpha\Phi^\alpha+\tfrac{1}{4}\kappa\epsilon^{\alpha\beta\gamma\rho}F_{\alpha\beta}F_{\gamma\rho}$$

yields the equations

$$\mathrm{rot}\,F_{\alpha\beta}+\kappa\,\mathrm{div}\,F_{\alpha\beta}=0,\quad \mathrm{div}\,F_{\alpha\beta}-2\kappa\,\mathrm{rot}\,F_{\alpha\beta}=j_\alpha.$$

Here κ is any constant. These equations are Maxwellian if $\kappa=0$ or $j_\alpha=0$, but not otherwise. The author mentions the connection of his theory with those of Bopp, Born, Mie, Proca, and Yukawa. *J. L. Synge* (Dublin).

Ashour, A. A. The induction of electric currents in a uniformly conducting circular disk by the sudden creation of magnetic poles. Quart. J. Mech. Appl. Math. 5, 379–384 (1952).

The problem of the induction and decay of electric currents which are induced in a circular disc of uniform surface resistance σ and radius a by the sudden introduction of a single magnetic pole is treated. Using toroidal coordinates, a double space of two regions is constructed in the manner of Sommerfeld's theory. The Riemann potential for a single pole is obtained and Jeans's treatment of uniform finite plane current sheets [Jeans, Proc. London Math. Soc. 31, 151–169 (1899)] is applied. *F. Oberhettinger.*

Marx, G. Das elektromagnetische Feld in bewegten anisotropen Medien. Acta Phys. Acad. Sci. Hungar. 3, 75–94 (1953). (Russian summary)

The relativistic form of Maxwell's field equations are extended to include anisotropic media for which both the dielectric constant and the permeability are given as tensors. This is applied to compute the polarization of an anisotropic parallelepiped moving in an arbitrary direction and an experiment is suggested to test the theoretically expected surface charge distributions. The author computes also the Lagrange function for the general electromagnetic field as well as the stress tensor. It is shown that for the anisotropic medium, the stress tensor based on Abraham's theory for isotropic media differs from that based on Minkowski's theory. No critical evaluation is offered.

E. Weber (Brooklyn, N. Y.).

Cade, R. Some electrostatic and steady-current problems involving anisotropic bodies. Proc. Phys. Soc. Sect. B. 66, 557–569 (1953).

This paper gives the results of extensive calculations of potential and forces and torques on anisotropic bodies both for dielectric and conducting solid spheres and spheroids. The complete mathematical steps are contained in a dissertation submitted in 1952 to the University of London. For the force actions, use is made of the Maxwell stresses which have been demonstrated by W. F. Brown, Jr. [Amer. J. Phys. 19, 290–304 (1951)] to be valid for anisotropic media.

The first problem concerns the electrostatic potential solution for a sphere of uniform, linear, anisotropic solid dielectric with symmetric dielectric tensor $\epsilon_{\alpha\beta}$ immersed in a fluid dielectric of uniform dielectric constant ϵ_0 which, before immersion, was influenced by a uniform electric field E_0. The solution indicates a uniform field E_i inside the sphere, but the direction of this internal field is generally different from that of E_0, except when the latter was oriented along the principal axis of $\epsilon_{\alpha\beta}$. The outside field is that of a doublet superimposed on the original field; however, the direction of the doublet is generally different from both E_0 and E_i except when E_0 is along the principal axis of $\epsilon_{\alpha\beta}$ when all three directions coincide. There is no force on the sphere, only a torque exists, which is characteristic for the anisotropic dielectric. For the conducting anisotropic sphere the same general characteristics hold, except that a surface charge appears because of the joint action of conductivity and dielectric constant.

The potential solution for the anisotropic dielectric oblate and prolate spheroids in spheroidal coordinates involves the spheroidal harmonics with coefficients which are very complex functions of the matrix elements $\epsilon_{\alpha\beta}$. Taking one of the principal axes coincident with the axis of revolution leads to more manageable forms and in this case the stress tensor is computed. Again, there is no resultant force, but a couple exists for which explicit expressions are given. The solution for the corresponding conducting cases are presumed to be entirely original and it is suggested that new and significant measurements might be feasible which might lead to the determination of crystalline conductivities in a new way.

It is also pointed out that the theory presented might explain inconsistencies between experimental evidence and former incorrect theories. *E. Weber.*

Pogorzelski, W. Contribution à la théorie du champ électromagnétique. Prace Mat.-Fiz. 48, 53–58 (1952).

The electric intensity is given for $t\geqq 0$ throughout a bounded empty region S. The problem of determining the magnetic intensity, given its initial normal components on the boundary of S, reduces to the determination of a vector given its curl and divergence in S and its normal components

on the boundary of S, which in turn reduces to known soluble problems of potential theory [cf. Aržanyh, Doklady Akad. Nauk SSSR (N.S.) **82**, 545–548 (1952); **85**, 55–58 (1952); these Rev. **14**, 46]. *F. V. Atkinson* (Ibadan).

*Pratelli, Aldo M. **Leggi integrali del campo di Proca-Yukawa nell'interpretazione di Bopp.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 550–558. Casa Editrice Perrella, Roma, 1953.

In Minkowski space-time, let $F_{\alpha\beta}$ be the 6-vector of electric force and magnetic induction; $f_{\alpha\beta}$ that of electric displacement and magnetic force; and $U_{\alpha\beta}$ the Proca-Yukawa 6-vector. In the theory of Bopp [Ann. Physik (5) **38**, 345–384 (1940); **42**, 573–608 (1942); these Rev. **2**, 336; **8**, 124], $F_{\alpha\beta}$, $f_{\alpha\beta}$ are linearly related, $f_{\alpha\beta}$ satisfies all eight of the Maxwell-Lorentz equations, and $U_{\alpha\beta}=f_{\alpha\beta}-F_{\alpha\beta}$. The author obtains various integral theorems. For example, if ω is a closed 2-surface in space-time,

$$\oint_\omega U_{\alpha\beta}dx^\alpha\wedge dx^\beta=0;\quad \oint_\omega F_{\alpha\beta}dx^\alpha\wedge dx^\beta=0,$$

and if σ is an open 3-surface bounded by ω,

$$\frac{1}{2}\oint_\omega U_{\alpha\beta}N^{\alpha\beta}d\omega=-\chi\int U_\alpha n^\alpha d\sigma+\int_\sigma J_\alpha n^\alpha d\sigma,$$

where U_α is the Proca-Yukawa 4-potential, χ is a scalar, J_α the charge-and-current 4-vector and $N^{\alpha\beta}$ the 6-vector normal to ω. Numerous formulae of this sort are obtained, the results being stated as flux-theorems concerning the various physical quantities involved. *H. S. Ruse.*

Balazs, N. L. The energy-momentum tensor of the electromagnetic field inside matter. Physical Rev. (2) **91**, 408–411 (1953).

Two different energy-momentum tensors have been proposed to describe the electromagnetic field inside matter. Abraham suggested a symmetric tensor, while Minkowski's is asymmetric. Von Laue [Z. Physik **128**, 387–394 (1950); these Rev. **12**, 377] has given arguments in favour of the asymmetric tensor. The present author, however, advances reasons for choosing the symmetric tensor. These are based upon an ideal experiment designed to show that only the symmetric tensor satisfies the momentum-conservation and centre-of-mass theorems simultaneously. *H. S. Ruse.*

Reza, Fazlollah. Deux théorèmes sur les dipôles électriques. C. R. Acad. Sci. Paris **237**, 429–430 (1953).

A brief note demonstrating that if $Z(p)=N(p)/D(p)$ represents the impedance of a finite lossless two-pole, the function $Z_n(p)=[d^nN(p)/dp^n]/[d^nD(p)/dp^n]$ will have the same character. If $Z(p)$ is the impedance of a two-pole composed of resistors and either capacitors or inductors but not both, then $Z_n(p)$ will again have the same character.

E. Weber (Brooklyn, N. Y.).

Quantum Mechanics

Fok, V. A. On the so-called ensembles in quantum mechanics. Vestnik Leningrad Univ. **1952**, no. 6, 67–73 (1952). (Russian)

This paper, written as a review of D. I. Blohincev's book "Foundations of quantum mechanics" [2nd ed., Gostehizdat, Moscow-Leningrad, 1949], criticises Blohincev for teaching that a wave-function is a description of a statistical ensemble and not a property of an individual object. The author explains clearly his own point of view concerning the philosophical meaning of quantum mechanics, a point of view with which most physicists nowadays will not seriously differ. His views will be of interest to those who are concerned with epistemological aspects of quantum mechanics. *F. J. Dyson* (Princeton, N. J.).

Bopp, Fritz. Ein statistisches Modell für den Grundprozess in der Quantentheorie der Teilchen. Z. Naturforschung **8a**, 228–233 (1953).

What is considered to be the fundamental process of the quantum theory of particles, viz. the presence or absence of a particle at a point in space, and the time dependence of this, is analysed on the basis of certain postulates.

C. Strachan (Aberdeen).

Volz, Helmut. Der korrespondenzmässige Zugang zur Quantentheorie der Wellenfelder. Z. Naturforschung **7a**, 70–75 (1952).

The commutation rules for a wave field are derived by using the analogue of the Bohr-Sommerfeld quantum condition. This method fails, of course, for the Fermi-Dirac case.

H. Feshbach (Cambridge, Mass.).

Umezawa, Hiroomi, and Takahashi, Yasushi. The general theory of the interaction representation. II. General fields and interactions. Progress Theoret. Physics **9**, 501–523 (1953).

This paper extends the results of the authors' earlier paper [same journal **9**, 14–32 (1953); these Rev. **15**, 83] to systems of fields with interactions containing higher derivatives of the field quantities. The authors' method is to take the field operators of the interaction representation to be of the conventional kind, satisfying field equations of first or second order. All the terms involving higher derivatives are put into the interaction part of the Lagrangian. This makes the formal construction of the interaction representation quite simple, and the equations are closely parallel to those of the authors' earlier paper. However, it is not clear to the reviewer that this interaction representation gives a correct description of the system. The extra degrees of freedom, which a field acquires when it satisfies a field equation with higher derivatives, are somehow suppressed and do not appear in the interaction representation. The authors observe that their method differs from that of Ostrogradski [see Pais and Uhlenbeck, Physical Rev. (2) **79**, 145–165 (1950); these Rev. **12**, 227], who quantized directly the field operators which satisfy higher-order equations, and obtained the expected extra degrees of freedom. The physical meaning of this difference is not discussed. *F. J. Dyson.*

Hamilton, J. Convergence in the intermediate representation. Proc. Cambridge Philos. Soc. **49**, 642–649 (1953).

The author claims to have proved that the power-series expansion of the field-operators in the "intermediate representation" introduced by the reviewer [Physical Rev. (2) **83**, 608–627 (1951); these Rev. **13**, 608] is convergent, neglecting certain terms which probably do not affect the convergence decisively. The proof is incorrect; the methods of Thirring [Helvetica Phys. Acta **26**, 33–52 (1953); these Rev. **14**, 708] can probably be adapted so as to prove that the power-series diverges.

The reviewer is to blame for having, in the paper cited, proclaimed his belief in the convergence, a belief which he abandoned some months later. The author's error lies in his equation (7)

$$\prod(2\Delta_j)^{-1} = (2^n n!\Gamma^n)^{-1}.$$

He bases his argument on the reviewer's analysis of Heisenberg operators in terms of "doubled graphs". His equation (7) holds if, as he explicitly assumes, the doubled lines in such a doubled graph form a single connected curve. However, the doubled lines in general form not a curve but a tree. For an average tree, the order of magnitude of the product on the left of equation (7) is much larger than $(n!)^{-1}$, and this destroys the convergence irreparably. The author has informed the reviewer that he is in agreement with this criticism. *F. J. Dyson* (Princeton, N. J.).

Petermann, A. Renormalisation dans les séries divergentes. Helvetica Phys. Acta **26**, 291–299 (1953).

In an earlier paper [Arch. Sci. Soc. Phys. Hist. Nat. Genève **6**, 5–23 (1953)] the author proved the divergence of perturbation expansions for a scalar meson theory with a non-linear interaction $\lambda\phi^3$, but without including the effects of mass and coupling-constant renormalization. In this paper he shows that the renormalizations will not change the situation, a conclusion reached independently by W. Thirring [Helvetica Phys. Acta **26**, 33–52 (1953); these Rev. **14**, 708]. He points out a fact which Thirring did not discuss, that the "renormalized coupling constant" may be defined in many alternative ways, no one of which has any superiority over the others. Thus if λ_0 is one choice of the renormalized coupling constant, we may equally well express the results of the theory formally in terms of λ_1 defined by

$$\lambda_1^2 = \lambda_0^2[1 + R_3\lambda_0^2 + R_4\lambda_0^4 + \cdots], \tag{1}$$

where $R_3, R_4, \cdots$ are arbitrarily chosen. If the results of the theory were expressible as convergent series in λ_0, then a substitution of the form (1), the series on the right of (1) being also convergent, would still yield series in λ_1^2 with a finite or infinite radius of convergence. However, the author's argument shows that this is not the case, and this makes the results of the theory highly ambiguous. In fact, a divergent series in λ_0^2 can be transformed by a substitution (1) into a series in λ_1^2, the first few terms of which can be chosen arbitrarily. This makes doubtful any attempt to attach physical significance to the first few terms of such a series expansion.

Only in quantum electrodynamics this difficulty does not arise, because in that case the "renormalized coupling constant" is the observed electronic charge and has a unique definition in terms of possible measurements.

F. J. Dyson (Princeton, N. J.).

Katayama, Yasuhisa. Theory of the interactions with higher derivatives and its application to the non-local interaction. Progress Theoret. Physics **10**, 31–56 (1953).

The author proposes a new quantization method for fields with an interaction containing higher order derivatives of the field variables. It is based on a general method for constructing a Hamiltonian in the interaction representation, proposed by Takahashi and Umezawa [same journal **9**, 14–32 (1953); these Rev. **15**, 83]. The Hamiltonian and the pairs of canonically conjugate variables are obtained in the form of power series in the coupling constant. For zero coupling the canonical variables reduce to their conventional expression for the free fields. Their number is always the same as for free fields, a feature known not to hold in general for the more conventional method of quantization based on classical mechanics. The present method avoids negative energies. The constructed Hamiltonian is used to find the S-matrix; the expression obtained is identical with the result given by the Yang-Feldman method [Physical Rev. (2) **79**, 972–978 (1950); these Rev. **12**, 569]. The extension to non-local interactions is briefly discussed.

L. Van Hove (Princeton, N. J.).

Salam, Abdus. Modified propagation functions in perturbation theory. Proc. Cambridge Philos. Soc. **49**, 638–641 (1953).

In most problems of field theory, transition amplitudes are calculated between states which differ from the vacuum by the presence of a small number of particles. In some special cases, one wishes to calculate transition amplitudes between states differing from a given continuous distribution of free particles by the absence or presence of a small number of particles. For a Fermi-Dirac field, the author shows this to be possible by simply modifying the definition of the Feynman propagation function. For a Bose field, after a similar modification of the propagation function, the author finds extra terms of complicated nature, $G(x)$ and $H(x)$ in his notation. It seems to the reviewer, however, that these terms should be inversely proportional to the volume of a large box containing the field, and should therefore be neglected. *L. Van Hove* (Princeton, N. J.).

Faure, Robert. Transformations conformes en mécanique ondulatoire. C. R. Acad. Sci. Paris **237**, 603–605 (1953).

Svenonius, Per. On the coherent scattering of light by an electrostatic field according to the Dirac hole theory. Ark. Fys. **6**, 269–277 (1953).

Second-order perturbation theory is used in discussing the interaction between photons, and particles in an electrostatic field: it is assumed that the potential of this field is weak and that the particles fill all the negative-energy states while all the others are empty. It is hoped to give later the results of the calculation. *C. Strachan* (Aberdeen).

Hoffmann, Banesh. The similarity theory of relativity and the Dirac-Schrödinger theory of electrons. II. Physical Rev. (2) **91**, 751–752 (1953).

The author's earlier paper [Physical Rev. (2) **89**, 52–59 (1953); these Rev. **14**, 821] is modified in order to base the theory on an integral which is invariant under similarity transformations. Jordan's theory [Ann. Physik (6) **1**, 219–228 (1947); these Rev. **9**, 538] is shown to be a particular case of the present theory. *A. J. Coleman.*

Valatin, Jean G. Sur l'interprétation des opérateurs de la théorie du positron. J. Phys. Radium (8) **12**, 607–615 (1951).

The effect of charge conjugation is discussed for the operators of the one-electron theory and the corresponding operators of the electron-positron field. Two formal rules of correspondence between one electron and electron-positron operators are defined, differing by their behavior under charge conjugation. They are illustrated by examples. An extended formulation of Furry's symmetry theorem [Physical Rev. (2) **51**, 125–129 (1937)] is given.

L. Van Hove (Princeton, N. J.).

Sengupta, N. D. On the solution of the equation of electron in the field of electromagnetic radiation. Bull. Calcutta Math. Soc. **44** (1952), 132–136 (1953).

The author obtains an exact solution of Dirac's equation for an electron in the quantized electromagnetic radiation field incident on the electron from all directions.

A. H. Taub (Urbana, Ill.).

van Loc, Phan. Sur le principe de Huygens en théorie de l'électron de Dirac. C. R. Acad. Sci. Paris **237**, 649–651 (1953).

A derivation of an expression for the Dirac wave function in terms of its values on a surface and over the volume at a retarded time. Essentially this amounts to a derivation of the Green's function for the Dirac equation.

H. Feshbach (Cambridge, Mass.).

Thermodynamics, Statistical Mechanics

Rosen, Philip. On variational principles for irreversible processes. J. Chem. Phys. **21**, 1220–1221 (1953).

The author observes that the Euler equation of

$$\delta\int\left[C_v T\frac{\partial T}{\partial t}-\frac{\sigma}{2}(\nabla T)^2\right]d\tau=0,$$

where the notations are classical and the statement itself is misprinted in the author's paper, is Fourier's equation for the diffusion of heat, provided all competing functions T have the same time derivative $\partial T/\partial t$. In a similar spirit he writes down for viscous incompressible fluids a "restricted variational principle similar to Onsager's," where now all competing velocity fields have the same acceleration field. In addition to these drastic restrictions, the author systematically throws away every surface integral he encounters because it "has no effect on the differential equation obtained from the variational principle." [The reviewer has not attempted to check his suspicion that the combination of conscious and unconscious limitations thus imposed by the author render the set of competing functions other than the solution in each case empty.] *C. Truesdell.*

Popoff, Kyrille. Sur la thermodynamique des processus irréversibles. Bŭlgar. Akad. Nauk Izvestiya Mat. Inst. **1**, 111–126 (1953). (Bulgarian. French summary)

Yamamoto, Tsunenobu. Statistical mechanics of general Brownian motions underlying irreversible processes. Progress Theoret. Physics **10**, 11–23 (1953).

A discussion is given of the physical basis of two postulates used by M. S. Green [J. Chem. Phys. **20**, 1281–1295 (1952); these Rev. **14**, 1048] in his theory of the statistical mechanics of time-dependent phenomena. These two postulates are: (1) the macro-observables or gross variables of a system are approximately one-valued integrals of the dynamical motion; and (2) the small temporal changes of these variables are described by a Markoff process. The author rederives the Fokker-Planck type equation given by Green without employing Green's assumptions. In order to do this, however, the author has made numerous other assumptions some of which are explicitly stated but several of which are erroneously portrayed as mathematical deductions. *G. Newell* (Providence, R. I.).

***Bird, R. B., Hirschfelder, J. O., and Curtiss, C. F. Survey of the equation of state and transport properties of gases and liquids.** Proceedings of the Third Midwestern Conference on Fluid Mechanics, University of Minnesota, March 23, 24, and 25, 1953, pp. 3–84. University of Minnesota, Minneapolis, Minn., 1953. $6.00.

Herpin, André. Contribution à l'étude de la théorie cinétique des solides. Ann. Physique (12) **7**, 91–139 (1952).

The author considers various physical consequences arising from the inclusion of small anharmonic energy terms in the equations of motion of an otherwise elastic solid. The model used for the elastic medium is that of Debye and Brillouin, a modified continuum model with dispersion.

The article includes detailed discussions of the model, the modes of vibration, and the interaction between the waves due to the anharmonic perturbation. The elastic waves are interpreted as a phonon gas in which collisions between phonons arise from the anharmonic interaction. The relaxation time of a phonon or elastic wave caused by these collisions is calculated as a function of temperature for various types of collisions such as those between transverse waves and longitudinal waves, transverse waves and transverse waves, etc. Finally from this, the thermal conductivity is calculated.

This reviewer had some difficulty in deciding if this paper contained any new results. Certainly much of this quite lengthy paper consists of a reformulation of old ideas arising primarily from the work of Debye (1914) and Peierls (1929). The exact references to these and many subsequent articles on this subject can be found in a review paper (published since this paper by Herpin) by Berman [Advances in Physics **2**, 103–140 (1953)]. *G. F. Newell* (Providence, R. I.).

Oguchi, Takehiko. Statistics of the three-dimensional ferromagnet. II, III. J. Phys. Soc. Japan **6**, 27–31, 31–35 (1951).

[For part I see same J. **5**, 75–81 (1950); these Rev. **13**, 196.] These papers are mainly concerned with the variation of propagation of order as a function of temperature in the three-dimensional (simple cubic lattice) Ising model of a ferromagnet. The problem is to find the influence of the spin state of a given lattice point on the probability distribution of various spin states at lattice points 1, 2, 3, $\cdots$ lattice distances away. At temperatures below the Curie temperature this influence extends over very large distances while above the Curie temperature it dies out very rapidly. The author uses the combinatorial method to investigate this problem. He considers the propagation of order associated with single spins and with pairs of spins. Both high and low temperature expansions are obtained for these effects.

E. W. Montroll (College Park, Md.).

ter Haar, D. Statistics of the three-dimensional ferromagnet. III. Some numerical results. Physica **19**, 611–614 (1953).

Further numerical results for the "variational method" described in a previous paper [Martin and ter Haar, Physica **18**, 569–581 (1952); these Rev. **14**, 522] are given. Thus the energy and specific heat are obtained for $J/kT=.02$, .05, .08, .10, .12, .15, .18, .20 and .22, and compared with the results obtained by Bethe and by Trefftz.

F. J. Murray (New York, N. Y.).